NPSOR-91-030

NAVAL POSTGRADUATE SCHOOL

Monterey, California

A KALMAN FILTER FOR A POISSON SERIES WITH COVARIATES AND LAPLACE APPROXIMATION INTEGRATION

Donald P. Gaver
Patricia A. Jacobs

September 1991

Prepared for:
Naval Postgraduate School
Monterey, CA 93943

NAVAL POSTGRADUATE SCHOOL,
MONTEREY, CALIFORNIA

Rear Admiral R. W. West, Jr.
Superintendent

Harrison Shull
Provost

This report was prepared in conjunction with research funded under the Naval Postgraduate School Research Council Research Program.

This report was prepared by:

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY 2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.
4. PERFORMING ORGANIZATION REPORT NUMBER(S) NPSOR-91-030	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Naval Postgraduate School 6b. OFFICE SYMBOL (*If applicable*) OR	7a. NAME OF MONITORING ORGANIZATION O&MN, Direct Funding
6c. ADDRESS (*City, State, and ZIP Code*) Monterey, CA 93943	7b. ADDRESS (*City, State, and ZIP Code*) Monterey, CA 93943-5006
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Naval Postgraduate School 8b. OFFICE SYMBOL (*If applicable*)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (*City, State, and ZIP Code*) Monterey, CA 93943	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO. / WORK UNIT ACCESSION NO.

11. TITLE (*Include Security Classification*)
A Kalman Filter for a Poisson Series with Covariates and Laplace Approximation Integration

12. PERSONAL AUTHOR(S)
Donald P. Gaver and Patricia A. Jacobs

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (*Year, month day*)	15. PAGE COUNT
Technical Report	FROM ____ TO ____	September, 1991	47

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (*Continue on reverse if necessary and identify by block number*)
FIELD	GROUP	SUB-GROUP	Poisson time series with covariates; hierarchical model; Kalman filter; Laplace method

19. ABSTRACT (*Continue on reverse if necessary and identify by block number*)

A hierarchical model for a Poisson time series is introduced. The model allows the mean or rate of the Poisson variables to vary slowly in time; it is modeled as the exponential of an AR/1 process. In addition the rate is influenced by a covariate. The Laplace method is used to recursively update some model parameter estimates. Frankly heuristic methods are explored to estimate other of the underlying parameters. The methodology is checked against simulated data with encouraging results.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT [x] UNCLASSIFIED/UNLIMITED [ ] SAME AS RPT. [ ] DTIC USERS	21. ABSTRACT SECURITY CLASSIATION UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL D. P. Gaver	22b. TELEPHONE (*Include Area Code*) (408) 646-2605 2c. OFFICE SYMBOL OR/Gv

DD Form 1473, JUN 86 *Previous editions are obsolete.* SECURITY CLASSIFICATION OF THIS PAGE

A KALMAN FILTER FOR A POISSON SERIES WITH COVARIATES AND LAPLACE-APPROXIMATION INTEGRATION

D. P. Gaver
P. A. Jacobs

0. INTRODUCTION

The Poisson model is an initial idealized, but plausible, off-the-shelf tool for representing point-process data of nearly any kind, cf. Feller (1966) and Cox and Lewis (1968). However, to be more descriptive, and even predictive, representation of non-homogeneity in space or time may be needed. For instance the occurrence of rare events in space, such as the occurrence of extreme heights, i.e. above a level of Arctic ice along a transsect or encounters with ice keels along a submarine track at constant (deep) depths, may well appear roughly Poisson. A better description of these events may require more detail than a simple mean or rate: some account of regional and seasonal variation could be needed for true accuracy; for instance the intervention of natural gaps along a path in Arctic ice will occur if the latter crosses **leads** (open water amidst an ice pack). For another example, demands for spare parts in a logistics system often are roughly Poisson, or compound Poisson, but with a mean or rate that changes erratically but slowly in time. Demands for communication and computer facilities exhibit a similar temporal pattern, and there are a great many other examples. To summarize, variation in a fundamental Poisson rate or mean is very often encountered in practice; it is possible that if this variation is slow-moving or persistent enough it can be exploited for short-term forecasting.

In this paper we study a model-based procedure for forecasting in the kind of environments described. The model introduced allows the mean or rate of the Poisson process to be itself a random process; the exponential of an AR/1 autoregressive process. In addition the rate is influenced by a covariate. We then recursively update the parameter estimates using an approximation based on the *Laplace method*; cf. de Bruijn (1958). The approach resembles that of Delampady, Yee and Zidek (1991), but frankly heuristic methods are used to estimate certain of the underlying parameters. The methodology is checked against simulated data with encouraging results.

1. FORMULATION

Consider the following Poissonian model for count data:

$$P\{Y_t = y_t | \mu_t, x_t, \beta, h_t, y_0, \ldots, y_{t-1}\} = \exp\left\{-h_t e^{x_t\beta + \mu_t}\right\} \frac{\left[h_t e^{x_t\beta + \mu_t}\right]^{y_t}}{y_t!} \tag{1.1}$$

where

$$\mu_t = \alpha\mu_{t-1} + \omega_t \tag{1.2}$$

with $\{\omega_t\}$ independent normal/Gaussian random variables with mean 0 and variance W_t independent of $\{\mu_i; i \le t-1\}$, $\{Y_i; i \le t-1\}$, $\{x_t\}$, $\{h_t\}$ and β. Another hierarchical time series model for count data can be found in Harvey et al. (1989).

The purpose of this paper is to suggest a Kalman filter-like procedure to produce successive estimates of β and μ_t as new data becomes available. The procedure is based on a Laplace approximation to an integral. A Bayesian approach to a similar problem is being investigated by Delampaday et al.

(1991). A Bayesian approach to time series can be found in West et al. (1989) and for a more recent computational approach in Carlin et al. (1991).

2. AN APPROXIMATE UPDATING PROCEDURE

Assume that the posterior distribution of $(\boldsymbol{\beta}, \mu_{t-1})$ given $\{y_i, i \leq t-1\}$ is bivariate normal with mean (b_{t-1}, m_{t-1}), $\mathrm{Var}[\boldsymbol{\beta}] = \tau_{t-1}$, $\mathrm{Var}[\mu_{t-1}] = C_{t-1}$ and $\mathrm{Corr}[\boldsymbol{\beta}, \mu_{t-1}] = \rho_{t-1}$.

Since it is known that

$$\mu_t = \alpha\mu_{t-1} + \omega_t,$$

the prior distribution of $(\boldsymbol{\beta}, \mu_t)$ is bivariate normal with mean $(b_{t-1}, \alpha m_{t-1})$, $\mathrm{Var}[\boldsymbol{\beta}] = \tau_{t-1}$, $\mathrm{Var}[\mu_t] = R_t = \alpha^2 C_{t-1} + W_t$ and $\mathrm{Corr}[\boldsymbol{\beta}, \mu_t] \equiv r_t = \alpha\rho_{t-1}\sqrt{C_{t-1}/R_t}$.

The forecast/prediction distribution of $\mathbf{Y}_t$ in terms of data up to t–1 and the covariate value at t is

$$P\{\mathbf{Y}_t = y_t | \mathbf{Y}_i = y_i, i \leq t-1, x_i, i \leq t, h_i, i \leq t\}$$

$$= \int \exp\left\{-e^{[x_t b + z]}h_t\right\} \frac{\left(e^{[x_t b + z]}h_t\right)^{y_t}}{y_t!} \frac{1}{2\pi\sqrt{\tau_{t-1}R_t}\left(1 - r_t^2\right)}$$

$$\times \exp\left\{-\frac{1}{2}\left(1 - r_t^2\right)^{-1}\left[\frac{\left(b - b_t^p\right)^2}{\tau_{t-1}} - \frac{2r_t\left(b - b_t^p\right)\left(z - m_t^p\right)}{\sqrt{\tau_{t-1}}\sqrt{R_t}} + \frac{\left(z - m_t^p\right)^2}{R_t}\right]\right\} dz\,db. \quad (2.1)$$

where $b_t^p = b_{t-1}$ and $m_t^p = \alpha m_{t-1}$.

We now approximate the integral by the Laplace method; cf. Easton (1991), Cox and Hinkley (1974), de Bruijn (1958). Let the exponent of the integrand be

$$g(b,z) = -e^{[x_t b+z]}h_t + y_t[x_t b + z] + y_t \ln h_t - \frac{1}{2}\left(1 - r_t^2\right)^{-1}$$

$$\times\left[\frac{\left(b - b_t^p\right)^2}{\tau_{t-1}} - \frac{2r_t}{\sqrt{\tau_{t-1}R_t}}\left(b - b_t^p\right)\left(z - m_t^p\right) + \frac{\left(z - m_t^p\right)^2}{R_t}\right] + K \tag{2.2}$$

where K is a constant.

$$\frac{\partial}{\partial z}g(b,z) = y_t - e^{x_t b+z}h_t - \frac{1}{1 - r_t^2}\left[\frac{\left(z - m_t^p\right)}{R_t} - r_t\frac{\left(b - b_t^p\right)}{\sqrt{\tau_{t-1}}\sqrt{R_t}}\right] \tag{2.3}$$

$$\frac{\partial^2}{\partial z^2}g(b,z) = -e^{(x_t b+z)}h_t - \frac{1}{1 - r_t^2}\frac{1}{R_t} \tag{2.4}$$

$$\frac{\partial}{\partial b}g(b,z) = \left(y_t - e^{x_t b+z}h_t\right)x_t - \left[1 - r_t^2\right]^{-1}\left[\frac{\left(b - b_t^p\right)}{\tau_{t-1}} - \frac{r_t\left(z - m_t^p\right)}{\sqrt{\tau_{t-1}}\sqrt{R_t}}\right] \tag{2.5}$$

$$\frac{\partial^2}{\partial b^2}g(b,z) = -e^{(x_t b+z)}h_t x_t^2 - \left[1 - r_t^2\right]^{-1}\frac{1}{\tau_{t-1}} \tag{2.6}$$

$$\frac{\partial^2}{\partial b\partial z}g(b,z) = -e^{(x_t b+z)}h_t x_t + (r_t)\left(1 - r_t^2\right)^{-1}(\tau_{t-1}R_t)^{-0.5}. \tag{2.7}$$

Use a Newton procedure to solve the system of equations

$$\begin{aligned} 0 &= \frac{\partial}{\partial z}g \\ 0 &= \frac{\partial}{\partial b}g \end{aligned} \tag{2.8}$$

for m_t and b_t. Solve for τ_t, C_t and ρ_t using the relation

$$\begin{bmatrix} \tau_t & \rho_t\sqrt{C_t\tau_t} \\ \rho_t\sqrt{C_t\tau_t} & C_t \end{bmatrix} = -\begin{bmatrix} \left.\frac{\partial^2}{\partial b^2} g\right|_{m_t,b_t} & \left.\frac{\partial^2}{\partial b\partial z} g\right|_{m_t,b_t} \\ \left.\frac{\partial^2}{\partial b\partial z} g\right|_{m_t,b_t} & \left.\frac{\partial^2}{\partial z^2} g\right|_{m_t,b_t} \end{bmatrix}^{-1}. \tag{2.9}$$

The posterior distribution for (β,μ_t) given y_t, D_{t-1} is approximated by a bivariate normal distribution with mean (b_t, m_t) and variance-covariance matrix

$$\Sigma_t = \begin{bmatrix} \tau_t & \rho_t\sqrt{C_t\tau_t} \\ \rho_t\sqrt{C_t\tau_t} & C_t \end{bmatrix}. \tag{2.10}$$

2.1 Summary of the Newton Procedure

0. Start with estimates of the parameters of the prior bivariate distribution of (β,μ_t); that is, estimates of the mean $(b_{t-1}, \alpha m_{t-1})$, $\mathrm{Corr}(\beta,\mu_t) = r_t$, $\mathrm{Var}(\beta) = \tau_{t-1}$ $\mathrm{Var}[\mu_t] = R_t$. Set $b_t^0 = b_{t-1}$, $m_t^0 = \alpha m_{t-1}$.

1. Solve the system of linear equations (in b and m)

$$0 = \frac{\partial}{\partial z} g(b_t^0, m_t^0) + \frac{\partial^2}{\partial b\partial z} g(b_t^0, m_t^0)[b - b_t^0] + \frac{\partial^2}{\partial z^2} g(b_t^0, m_t^0)[m - m_t^0]$$

$$0 = \frac{\partial}{\partial b} g(b_t^0, m_t^0) + \frac{\partial^2}{\partial b^2} g(b_t^0, m_t^0)[b - b_t^0] + \frac{\partial^2}{\partial z\partial b} g(b_t^0, m_t^0)[m - m_t^0]$$

for b, m.

2. If $\max\left[\left(\left|b_t^0 - b\right|/b\right), \left(\left|m_t^0 - m\right|/m\right)\right] < 0.001$, set $m_t = m$ and $b_t = b$ and go to 3. Otherwise set $b_t^0 = b$ and $m_t^0 = m$ and return to Step 1 unless Step 1 has been returned to 49 times in which case set $m_t = m_t^0$ and $b_t = b_t^0$; go to 3.

3. Return (b_t, m_t) as the estimate of the posterior mean of the bivariate normal distribution of (β,μ_t).

4. To find the variance-covariance matrix of the posterior distribution of (β,μ_t) evaluate

$$-\begin{bmatrix} \frac{\partial^2}{\partial b^2} g(b_t,m_t) & \frac{\partial^2}{\partial z \partial b} g(b_t,m_t) \\ \frac{\partial^2}{\partial z \partial b} g(b_t,m_t) & \frac{\partial^2}{\partial z^2} g(b_t,m_t) \end{bmatrix}^{-1};$$

set it equal to

$$\begin{bmatrix} \tau_t & \rho_t \sqrt{\tau_t}\sqrt{C_t} \\ \rho_t \sqrt{\tau_t}\sqrt{C_t} & C_t \end{bmatrix}$$

and solve for τ_t, ρ_t and C_t.

2.2 Summary of Kalman Procedure

In summary the approximate Kalman procedure is as follows

0. Start with the parameters of the (approximate) posterior bivariate normal distribution of (β,μ_{t-1}) having mean (b_{t-1}, m_{t-1}), $\mathrm{Corr}(\beta,\mu_{t-1}) = \rho_{t-1}$, $\mathrm{Var}[\beta] = \tau_{t-1}$ $\mathrm{Var}[\mu_{t-1}] = C_{t-1}$.

1. Update $\mathrm{Corr}(\beta,\mu_{t-1})$ and $\mathrm{Var}[\mu_{t-1}]$ using (1.2) to obtain the prior bivariate normal distribution of (β,μ_t) having mean $(b_{t-1}, \alpha m_{t-1})$, $\mathrm{Var}[\beta] = \tau_{t-1}$, $\mathrm{Var}[\mu_t] \equiv R_t = \alpha^2 C_{t-1} + W_t$, $\mathrm{Corr}(\beta,\mu_t) \equiv r_t = \alpha\rho_{t-1}\sqrt{C_{t-1}/R_t}$.

2. Observe the Poisson count y_t.

3. Invoke the Newton procedure of Section 2.1 to obtain estimates of the parameters of the approximate posterior distribution of (β,μ_t) given past observations and the new observation y_t.

To obtain moments for the Poisson mean $\boldsymbol{\lambda}_t = h_t \exp\{x_t\boldsymbol{\beta}+\boldsymbol{\mu}_t\}$ note that since the distribution of $(\boldsymbol{\beta}_t, \boldsymbol{\mu}_t)$ is being approximated by a bivariate normal distribution, the posterior moments of $\boldsymbol{\lambda}_t$ are approximately

$$E[\boldsymbol{\lambda}_t] \approx \exp\left\{x_t b_t + m_t + \frac{1}{2}\left[x_t^2 \tau_t + C_t + 2x_t \rho_t \sqrt{\tau_t}\sqrt{C_t}\right]\right\} h_t$$

$$\mathrm{Var}[\boldsymbol{\lambda}_t] \approx h_t^2 \exp\left\{2(x_t b_t + m_t) + \left[x_t^2 \tau_t + C_t + 2x_t \rho_t \sqrt{\tau_t}\sqrt{C_t}\right]\right\}$$

$$\times\left[\exp\left\{x_t^2 \tau_t + C_t + 2x_t \rho_t \sqrt{\tau_t}\sqrt{C_t}\right\} - 1\right].$$

The estimate of $\boldsymbol{\lambda}_t$ is $\hat{\lambda}_t = e^{b_t x_t + m_t} h_t$.

2.3 A Simulation Example

In the example $\boldsymbol{\beta} = 0.5$, $\{x_t\}$ takes the values $\{0.25, .5, 1\}$ over and over; $h_t \equiv 1$. $\{\boldsymbol{\omega}_t\}$ are iid normal with mean 0 and variance 0.25 and $\alpha = 0.5$. Given $\boldsymbol{\beta}$ and $\boldsymbol{\mu}_t$, Y_t has a Poisson distribution with mean $\boldsymbol{\lambda}_t = e^{x_t\boldsymbol{\beta}+\boldsymbol{\mu}_t}$. The simulation starts with $\boldsymbol{\mu}_0$ drawn from the stationary distribution of $\{\boldsymbol{\mu}_t\}$ a normal distribution with mean 0 and variance $W/(1-\alpha^2) = \frac{1}{3}$. Successive $\boldsymbol{\mu}_t$ are computed as

$$\boldsymbol{\mu}_t = \alpha\boldsymbol{\mu}_{t-1} + \boldsymbol{\omega}_t;$$

the $\boldsymbol{\lambda}_t$ is computed and a Poisson random number with mean $\boldsymbol{\lambda}_t$ is then generated. The simulation data are the Poisson counts and $\{x_t\}$; $t = 1, \ldots, 100$. The random numbers were generated using LLRANDOM II; cf. Lewis et al. (1981).

At time 0, the Newton procedure is initialized at $\mu_0 = 0$, $\beta = 0$, $\rho_0 = r_0 = 0$, $\alpha = 0.5$, $W = 0.25$, $C_0 = 0$, $r_1 = 0.25$, $\tau_0 = 1$; note the α and W are assumed known. The data point y_1 is observed and the Newton procedure is used to find the posterior moments $[m_1, b_1, \rho_1, C_1, \tau_1]$.

The da a point y_2 is the observed and the Newton procedure is started with $[m_1, b_1, r_1, C_1 + \frac{1}{4}, \tau_1]$ and used to find $[m_2, b_2, \rho_2, C_2, \tau_2]$, etc.

Results of the simulation appear in Figures 1-3. In Figure 1 the count data y_t appears along with the true λ_t (dotted line) and the estimated λ_t (solid line). In Figure 2 the true value of μ_t (dotted line) and estimated value of μ_t (solid line) appear. The estimated values of the standard deviation of $\hat{\mu}_t$, $\sqrt{C_t}$, also appear in Figure 2. The estimated value of β, β's estimated standard deviation $\sqrt{\tau_t}$, and the estimated value of ρ_t appear in Figure 3. Figures 1-3 indicate that the procedure performs satisfactorily. The apparent oscillation in some of the figures, (particularly $\hat{\rho}_t$), is due to the cyclic nature of $\{x_t\}$. Not surprisingly, the estimated values of λ_t and μ_t are less variable than the true values. They appear to be practically acceptable.

3. APPROXIMATE KALMAN PROCEDURES WHICH INCORPORATE ESTIMATES OF THE AUTOREGRESSIVE PARAMETERS α AND W: NAIVE MOMENT ESTIMATORS

In this section we will assume that $\{\omega_t\}$ are independent identically distributed normal/Gaussian random variables with mean 0 and variance W with ω_t independent of $\{\mu_i; i \le t-1\}$, $\{Y_i; i \le t-1\}$, $\{h_t\}$ and β.

3.1 Estimates of the Autoregressive Parameters α and W

If $\{\mu_t; t \le T\}$ were observable, then one could estimate α and W by maximum likelihood; that is, the likelihood function is

$$L(W,\alpha) = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi W}} \exp\left\{-\frac{1}{2}(\mu_t - \alpha\mu_{t-1})^2 \frac{1}{W}\right\} \tag{3.1}$$

or

$$\ln L(W,\alpha) \equiv l(W,\alpha) = \frac{1}{2}\sum_{t=1}^{T}(\mu_t - \alpha\mu_{t-1})^2 \frac{1}{W}$$

$$- \frac{T}{2}\ln W - \frac{T}{2}\ln 2\pi. \tag{3.2}$$

Now

$$\frac{\partial l}{\partial \alpha} = -\sum_{t=1}^{T}(\mu_t - \alpha\mu_{t-1})\mu_{t-1}\frac{1}{W} = 0 \tag{3.3}$$

gives

$$\hat{\alpha}(T) = \sum_{t=1}^{T}\mu_t\mu_{t-1} \Big/ \sum_{t=1}^{T}\mu_{t-1}^2 \tag{3.4}$$

and

$$\frac{\partial l}{\partial W} = \frac{1}{2}\frac{1}{W^2}\sum_{t=1}^{T}(\mu_t - \alpha\mu_{t-1})^2 - \frac{T}{2}\frac{1}{W} = 0 \tag{3.5}$$

from which

$$\hat{W}(T) = \frac{1}{T}\sum_{t=1}^{T}(\mu_t - \hat{\alpha}(T)\mu_{t-1})^2. \tag{3.6}$$

Unfortunately $\boldsymbol{\mu}_t$ is not observable. One possible estimate of $\boldsymbol{\mu}_t$ is the posterior mean m_t of subsections (2.1) and (2.2). In this case the corresponding estimators of α and W are

$$\hat{\alpha}_m(T) = \sum_{t=1}^{T}m_t m_{t-1} \Big/ \sum_{t=1}^{T}m_{t-1}^2 \tag{3.7}$$

$$\hat{W}_m(T) = \frac{1}{T}\sum_{t=1}^{T}(m_t - \hat{\alpha}_m(T)m_{t-1})^2. \tag{3.8}$$

Another estimate of μ_t is

$$\hat{\mu}_t(y) = \ln\left[y_t + \frac{1}{2}\right] - x_t b_t \tag{3.9}$$

where b_t is the mean of the approximate posterior distribution of $\boldsymbol{\beta}$ given $\{y_s; s \le t\}$. The corresponding estimates of α and W are

$$\hat{\alpha}_y(T) = \sum_{t=1}^{T} (\hat{\mu}_t(y)\hat{\mu}_{t-1}(y)) / \sum_{t=1}^{T} \hat{\mu}_t(y)^2 \tag{3.10}$$

and

$$\hat{W}_y(T) = \frac{1}{T}\sum_{t=1}^{T} \left(\hat{\mu}_t(y) - \hat{\alpha}_y(T)\hat{\mu}_{t-1}(y)\right)^2 . \tag{3.11}$$

The estimates (3.7) – (3.11) can be recomputed at every time T. Other similar estimates can be obtained by not recomputing the estimates at every time T; for example, choose an integer $\delta > 1$, put

$$\hat{\alpha}(t;\delta) = \hat{\alpha}((n-1)\delta;\delta) \tag{3.12}$$

and

$$\hat{W}(t;\delta) = \hat{W}((n-1)\delta;\delta) \tag{3.13}$$

if $(n-1)\delta < t < n\delta$;

$$\hat{\alpha}(n\delta;\delta) = \sum_{t=1}^{n\delta} \hat{\mu}_t\hat{\mu}_{t-1} / \sum_{t=1}^{n\delta} \hat{\mu}_{t-1}^2 \tag{3.14}$$

$$\hat{W}(n\delta;\delta) = \frac{1}{n\delta}\sum_{t=1}^{n\delta} (\hat{\mu}_t - \hat{\alpha}(n\delta;\delta)\hat{\mu}_{t-1})^2 \tag{3.15}$$

where $\hat{\mu}_t$ is an estimate of μ_t. Another possibility is to use a window of times to compute estimates of α and W.

The following is a summary of the Kalman procedure of Section 2 with the addition of estimation of α and W.

3.2 Summary of the Kalman Procedure with Estimation of α and W

0. Start with the parameters of the (approximate) posterior bivariate normal distribution of (β, μ_{t-1}) having mean (b_{t-1}, m_{t-1}), $\mathrm{Corr}(\beta,\mu_{t-1}) = \rho_{t-1}$, $\mathrm{Var}[\beta] = \tau_{t-1}$, $\mathrm{Var}[\mu_{t-1}] = C_{t-1}$.
1. Update $\mathrm{Corr}(\beta, \mu_{t-1})$ and $\mathrm{Var}[\mu_{t-1}]$ using (1.2) and the current estimates of α and W to obtain the prior bivariate normal distribution of (β, μ_t) having mean $(b_{t-1}, \hat{\alpha}_{t-1} m_{t-1})$, $\mathrm{Var}[\beta] = \tau_{t-1}$, $\mathrm{Var}[\mu_t] \equiv R_t = \hat{\alpha}_{t-1}^2 C_{t-1} + \hat{W}_{t-1}$, $\mathrm{Corr}(\beta, \mu_t) = \hat{\alpha}_{t-1}\rho_{t-1}\sqrt{C_{t-1}/R_t}$.
2. Observe the Poisson count y_t.
3. Invoke the Newton procedure of Subsection 2.1 to obtain estimates of the parameters of the posterior bivariate normal distribution of (β, μ_t) given past observations and the new observation y_t.
4. Compute new estimates of α and W: $\hat{\alpha}_t, \hat{W}_t$.
5. Return to 1.

3.3 Results of Simulation Experiments

Various simulation experiments were carried out using the procedures outlined in this section to estimate α and W. One striking phenomenon was that if the count data has a long run of zeroes, then the Kalman procedure reacts by estimating $\alpha > 1$ and trying to estimate μ_t to be a negative number large in absolute value. This tendency caused the numerical Newton procedure discussed in Section 2 to become unstable, and rendered the predicted value of λ_t very slow to respond to positive counts when they did eventually occur. This behavior was ameliorated by arbitrarily putting a lower bound of -2 on $\hat{\mu}_t$. A run of zero counts could also make the estimate of W become close to zero. A small value of W makes the Kalman filter unresponsive to count changes. This behavior was mitigated by recomputing

estimates of α and W every $\delta = 10$ time units rather than at every time, and using all previous times t as in (3.7)–(3.8). The estimates of α and W were also not computed if the last 10 observed counts were zero. Further a lower bound of 0.1 was arbitrarily set on estimates of W. Such heuristics are not claimed to be optimal, but are needed to allow the procedures to behave suitably. A search for more systematic procedures can be carried out in future work.

The following are empirical observations. The estimates of α and W using $\hat{\mu}(t) = m_t$ tends to produce practically acceptable estimates of α but underestimates W—it tends to make it very small. This behavior is not unexpected since m_t can be thought of as a smoothed estimate of the μ_t and so will have a smaller variance than μ_t. This underestimation of W is not improved by using $\hat{\mu}_t = \ln\left[y_t + \frac{1}{2}\right]$. Neither does using "windows" of length $\bar{\delta}$ seem to solve the problem. Other procedures for estimating α and W are explored in the following two sections.

4. ESTIMATION OF α AND W FOR THE POISSON/NORMAL MODEL: THE FREEMAN-TUKEY TRANSFORMATION

This section reports another possible approach to the estimation of the autoregressive parameters for a Poisson/normal model.

The model is as before

$$\mu_{t+1} = \alpha\mu_t + \omega_{t+1} \tag{4.1}$$

where $\{\omega_t\}$ are iid normal with mean 0 and variance W;

$$P\{Y_t = y|\mu_t, x_t\} = \exp\left\{-e^{\mu_t + \beta x_t} h_t\right\} \frac{\left[e^{\mu_t + \beta x_t} h_t\right]^y}{y!} . \tag{4.2}$$

For simplicity of notation we will assume $h_t \equiv 1$.

If W and α are known, then an approximate Kalman procedure has been given previously. The issue here is the estimation of α and W.

Let

$$g(x) = \sqrt{x} + \sqrt{x+1} \tag{4.3}$$

and

$$\mathbf{Z}_t = g(\mathbf{Y}_t). \tag{4.4}$$

The conditional distribution of $\mathbf{Z}_t$ given $\boldsymbol{\mu}_t$ is approximately normal with mean $k(\boldsymbol{\mu}_t + x_t\boldsymbol{\beta})$, where $k(x) = \left[4e^x + 1\right]^{0.5}$, and variance 1; c.f. Freeman and Tukey (1949, 1950) and Bishop, Fienberg and Holland (1975). Thus if $\{\mathbf{V}_{t+1}\}$ are iid standard normal, given $\mathbf{D}_t = (\mathbf{Y}_0, \mathbf{Y}_1, \ldots, \mathbf{Y}_t)$

$$\mathbf{Z}_{t+1} \overset{d}{\approx} k(\boldsymbol{\mu}_{t+1} + x_{t+1}\boldsymbol{\beta}) + \mathbf{V}_{t+1}$$

$$= k(\alpha\boldsymbol{\mu}_t + \boldsymbol{\omega}_{t+1} + x_{t+1}\boldsymbol{\beta}) + \mathbf{V}_{t+1}$$

$$= k\left(\alpha m_t + \alpha(\boldsymbol{\mu}_t - m_t) + \boldsymbol{\omega}_{t+1} + x_{t+1}b_t + x_{t+1}(\boldsymbol{\beta} - b_t)\right) + \mathbf{V}_{t+1}$$

$$\approx k(\alpha m_t + x_{t+1}b_t) + k'(\alpha m_t + x_{t+1}b_t)\left[\alpha(\boldsymbol{\mu}_t - m_t) + x_{t+1}(\boldsymbol{\beta} - b_t) + \boldsymbol{\omega}_{t+1}\right] + \mathbf{V}_{t+1}. \tag{4.5}$$

Hence,

$$E\left[\mathbf{Z}_{t+1} \middle| \mathbf{D}_t\right] \approx k(\alpha m_t + x_{t+1}b_t) \tag{4.6}$$

$$\mathrm{Var}\left[\mathbf{Z}_{t+1} \middle| \mathbf{D}_t\right] \approx \left[k'(\alpha m_t + x_{t+1}b_t)\right]^2\left[\alpha^2 C_t + x_{t+1}^2\tau_t + 2\alpha x_{t+1}\rho_t\sqrt{C_t}\sqrt{\tau_t} + W\right] + 1. \tag{4.7}$$

An approximate joint distribution of the transformed observations is

$$P\{\mathbf{Z}_1 \in dz_1, \mathbf{Z}_2 \in dz_2, \ldots, \mathbf{Z}_{T+1} \in dz_{T+1}\}$$

$$= P\{\mathbf{Z}_1 \in dz_1\} P\{\mathbf{Z}_2 \in dz_2 | \mathbf{Z}_1 = z_1\} \times \ldots \times P\{\mathbf{Z}_{T+1} \in dz_{T+1} | \mathbf{Z}_1 = z_1, \ldots, \mathbf{Z}_T = z_T\}$$

$$= P\{\mathbf{Y}_1 = dy_1\} P\{\mathbf{Z}_2 \in dz_2 | \mathbf{Y}_1 = y_1\} \times \ldots \times P\{\mathbf{Z}_{T+1} \in dz_{T+1} | \mathbf{Y}_1 = y_1, \ldots, \mathbf{Y}_T = y_T\}$$

$$\approx P\{\mathbf{Y}_1 = y_1\} \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi} f_t(\alpha, W)^{0.5}} \exp\left\{-\frac{1}{2}[z_{t+1} - k(\alpha m_t + x_{t+1} b_t)]^2 / f_t(\alpha, W)\right\} \quad (4.8)$$

where

$$f_t(\alpha, W) = \left[\alpha^2 C_t + x_{t+1}^2 \tau_t + 2\alpha\rho_t x_{t+1} \sqrt{C_t}\sqrt{\tau_t} + W\right] [k'(\alpha m_t + x_{t+1} b_t)]^2 + 1$$

Hence an approximate ln-likelihood function is (up to addition of constants),

$$l(\alpha, W) = \sum_{t=1}^{T} -\frac{1}{2} \ln f_t(\alpha, W) - \frac{1}{2}(z_{t+1} - k(\alpha m_t + x_{t+1} b_t))^2 f_t(\alpha, W)^{-1}. \quad (4.9)$$

Note that

$$k(x) = \left[4e^x + 1\right]^{0.5};$$

$$k'(x) = \frac{1}{2}\left[4e^x + 1\right]^{-0.5} 4e^x = \frac{2e^x}{\sqrt{4e^x + 1}};$$

$$k''(x) = \frac{2e^x\left[2e^x + 1\right]}{\left[4e^x + 1\right]^{1.5}}. \quad (4.10)$$

Further,

$$\frac{\partial}{\partial W} f_t(\alpha,W) = \left[k'(\alpha m_t + x_{t+1}b_t)\right]^2 \tag{4.11}$$

$$\frac{\partial}{\partial \alpha} f_t(\alpha,W) = \left[2\alpha C_t + 2x_{t+1}\rho_t\sqrt{C_t}\sqrt{\tau_t}\right]\left[k'(\alpha m_t + x_{t+1}b_t)\right]^2$$

$$+\left[\alpha^2 C_t + x_{t+1}^2\tau_t + 2\alpha\rho_t x_{t+1}\sqrt{C_t}\sqrt{\tau_t} + W\right]2\left[k'(\alpha m_t + x_{t+1}b_t)\right]$$

$$\times k''(\alpha m_t + x_{t+1}b_t)m_t. \tag{4.12}$$

Differentiating (4.9) with respect to α and W we obtain

$$\frac{\partial l}{\partial \alpha} = \sum_{t=1}^{T} -\frac{1}{2}\frac{\frac{\partial}{\partial \alpha}f_t(\alpha,w)}{f_t(\alpha,w)}\left[1 - \frac{\left[z_{t+1} - k(\alpha m_t + x_{t+1}b_t)\right]^2}{f_t(\alpha,W)}\right] \tag{4.13}$$

$$+(z_{t+1} - k(\alpha m_t + x_{t+1}b_t))k'(\alpha m_t + x_{t+1}b_t)m_t f_t(\alpha,W)^{-1}$$

$$\frac{\partial l}{\partial W} = \sum_{t=1}^{T} -\frac{1}{2}\frac{1}{f_t(\alpha,W)}\frac{\partial}{\partial W}f_t(\alpha,W) + \frac{1}{2}\left[z_{t+1} - k(\alpha m_t + x_{t+1}b_t)\right]^2 \frac{\frac{\partial}{\partial W}f_t(\alpha,W)}{f_t(\alpha,W)^2} \tag{4.14}$$

$$= \sum_{t=1}^{T} -\frac{1}{2}\frac{1}{f_t(\alpha,W)}\frac{\partial}{\partial W}f_t(\alpha,W)\left[1 - \frac{\left[z_{t+1} - k(\alpha m_t + x_{t+1}b_t)\right]^2}{f_t(\alpha,W)}\right]$$

$$E\left[\frac{\partial^2 l}{\partial \alpha^2}\right] = -\sum_{t=1}^{T}\left\{\frac{1}{2}\left[\frac{\frac{\partial}{\partial \alpha}f_t(\alpha,W)}{f_t(\alpha,W)}\right]^2 + \frac{\left[k'(\alpha m_t + x_{t+1}b_t)m_t\right]^2}{f_t(\alpha,W)}\right\}. \tag{4.15}$$

$$E\left[\frac{\partial^2 l}{\partial^2 W}\right] = -\sum_{t=1}^{T}\frac{1}{2}\left[\frac{\frac{\partial}{\partial W}f_t(\alpha,W)}{f_t(\alpha,W)}\right]^2. \tag{4.16}$$

$$E\left[\frac{\partial^2 l}{\partial\alpha\partial W}\right] = -\sum_{t=1}^{T}\frac{1}{2}\left[\frac{\frac{\partial}{\partial W}f_t(\alpha,W)\frac{\partial}{\partial\alpha}f_t(\alpha,W)}{f_t(\alpha,W)^2}\right] \tag{4.17}$$

To avoid difficulties with a Newton procedure involving the restricted range of $W > 0$, we will reparameterize; $W = e^{\gamma}$. In this case

$$\frac{\partial l}{\partial\gamma} = \frac{\partial l}{\partial W}e^{\gamma}$$

$$E\left[\frac{\partial^2 l}{\partial\gamma^2}\right] = E\left[\frac{\partial^2 l}{\partial W^2}\right]e^{2\gamma}$$

$$E\left[\frac{\partial^2 l}{\partial\alpha\partial\gamma}\right] = E\left[\frac{\partial^2 l}{\partial W\partial\alpha}\right]e^{\gamma}$$

and W is replaced by e^{γ} in all the expressions (4.12) – (4.17).

A Kalman procedure with this method of estimating α and W is similar to that of Section 3.2 except that Step 1 is replaced by the numerical solution of the system of equations,

$$\frac{\partial l}{\partial\alpha} = 0 \tag{4.18}$$

$$\frac{\partial l}{\partial\gamma} = 0 \tag{4.19}$$

by a Newton procedure using Fisher's scoring method. Set $\hat{W} = e^{\hat{\gamma}}$ where $\hat{\gamma}$ is the solution. Occasionally, there are numerical problems with the Newton procedure. In these cases search is used to find estimates of α and W.

A summary of the Newton (with default search) procedure is as follows. The procedure starts with the previously estimated α and γ called α_0 and γ_0.

0. Set $B = 0$.

1. Compute \f(∂l,∂α), \f(∂l,∂γ), E \b \bc \[(\f(∂^{2}l,∂α^2)), E $\left[\frac{\partial^2 l}{\partial \gamma^2}\right]$, and $E\left[\frac{\partial^2 l}{\partial\alpha\partial\gamma}\right]$ using α_0 and γ_0. If $\left|E\left[\frac{\partial^2 l}{\partial\alpha\partial\gamma}\right]\right| < 0.001$ or $E\left[\frac{\partial^2 l}{\partial \gamma^2}\right] < 0.0001$ go to 2′, otherwise go to 2.

2. Solve the system of linear equations (in terms α and γ)

$$0 = \frac{\partial l}{\partial \alpha} + E\left[\frac{\partial^2 l}{\partial \alpha^2}\right](\alpha - \alpha_0) + E\left[\frac{\partial^2 l}{\partial\alpha\partial\gamma}\right](\gamma - \gamma_0)$$

$$0 = \frac{\partial l}{\partial \gamma} + E\left[\frac{\partial^2 l}{\partial\alpha\partial\gamma}\right](\alpha - \alpha_0) + E\left[\frac{\partial^2 l}{\partial \gamma^2}\right](\gamma - \gamma_0)$$

for α_n and γ_n. Go to 3.

2′. Set $\gamma_n = \gamma_0$. If $\frac{\partial l}{\partial \alpha}$ is of the same sign for both the current and previous value of α, set

$$\alpha_n = \alpha_0 - \frac{\partial l}{\partial \alpha} \Big/ E\left[\frac{\partial^2 l}{\partial \alpha^2}\right]$$

and go to 3. If $\frac{\partial l}{\partial \alpha}$ is of different signs for the current and previous values for α, then a golden section search is used to solve the equation

$$0 = \frac{\partial l}{\partial \alpha};$$

set α_n equal to the first value of α for which $\left|\frac{\partial l}{\partial \alpha}\right| < 0.1$; if the number of iterations for this one-dimensional search is greater than 50, go to 4.

3. If $\max\left\{\left|\frac{\partial l}{\partial \alpha}\right|(\alpha_n), \left|\frac{\partial l}{\partial \alpha}\right|(\gamma_n)\right\} < 0.1$ stop and take α_n and γ_n as the estimated values.

3′. If $|\alpha_n| < 5$ and $\gamma_n < 5$, go to 5.

3″. If $|\alpha_n| > 5$ or $\gamma_n > 5$, let $B = B+1$. If $B = 1$, go to 4. If $B = 2$, go to 4′. If $B \geq 3$, to to 4″.

4. Do a two-dimensional search of the ln-likelihood function (4.9) parameterized in terms of α and W for a maximizing value. The grid for α is [–4,4] in steps of 0.2; the grid for W is [0,8] in steps of 0.2. If

$$\max\left\{\left|\frac{\partial l}{\partial \alpha}\right|(\alpha_n), \left|\frac{\partial l}{\partial \gamma}\right|(\ln W)\right\} < 0.1$$

return α_n and $\gamma_n = \ln W$ as the estimated values; otherwise set $\alpha_0 = \alpha_n$ and $\gamma_0 = \ln W$ and return to 1.

4′. Same as 4 except the grid for α is [–8,8] in steps of 0.1 and the grid for W is [0.1,8] in steps of 0.1 with the additional points 0.000001, 0.00001, 0.0001, 0.001, 0.01.

4″. γ is set equal to its previous value and a golden section search for the zero of $\frac{\partial l}{\partial \alpha}$ is done as in 2′.

5. Set $\alpha_o = \alpha_n$ and $\gamma_o = \gamma_n$ and return to 1 if the number of iterations is less than 50. If the number of iterations is greater than 50 set B = B + 1. If B = 1, go to 4. If B ≥ 6, return α_n and γ_n as the estimated values. If 1 < B ≤ 5, go to 6.

6. Compute $\left|\frac{\partial l}{\partial \alpha}(\alpha_n)\right|$ and $\left|\frac{\partial l}{\partial \gamma}(\gamma_n)\right|$. If $\left|\frac{\partial l}{\partial \alpha}(\alpha_n)\right| < \left|\frac{\partial l}{\partial \gamma}(\gamma_n)\right|$ go to 6′; otherwise go to 6″.

6′. Fix $\alpha_n = \alpha_0$ and do a search over the interval [-4B, 4B] for that γ for which

$$0 = \frac{\partial l}{\partial \gamma}.$$

Set γ_n equal to the first value of γ for which $\left|\frac{\partial l}{\partial \gamma}\right| < 0.1$; if the number of iterations is greater than 50, go to 4.

6″. Fix $\gamma_n = \gamma_0$ and do a search over the interval [–4B, 4B] for that α for which $0 = \frac{\partial l}{\partial \alpha}$; set α_n equal to the first value of α for which $\left|\frac{\partial l}{\partial \alpha}\right| < 0.1$; if the number is greater than 50, go to 4.

The following are empirical observations. The procedure involves a great deal of computing. For small times t, the estimate of W can be very close to zero; a lower bound of 0.1 was placed on the value of W that is input into the Kalman procedure. Further, a lower bound of −2 was also put on $\hat{\mu}_t$. The stopping criteria of $\max\left(\left|\frac{\partial l}{\partial \alpha}\right|, \left|\frac{\partial l}{\partial \gamma}\right|\right) < 0.1$ seems to be adequate; a smaller tolerance can result in the procedure becoming unstable and greatly increasing the computational effort.

5. ESTIMATION OF α AND W FOR THE POISSON/NORMAL MODEL: LOGARITHMIC TRANSFORMATION

Results of Lambert (1989) suggest that when count data arise from a mixture of Poisson distributions, then a smaller power transformation of the counts than the square root is needed to stabilize the variance of the count data. In particular, if the variability of the mixture is largish, then a log transformation may stabilize the variance. In this section simple procedures for estimating α and W based on a log transformation of the count data are described.

Suppose Y_t and μ_t are as in (1.1) and (1.2). Given μ_{t+1}, β, the first two terms of a Taylor expansion yield

$$\ln Y_{t+1} \approx \ln h_{t+1} + x_{t+1}\beta + \mu_{t+1} + [h_t \exp\{x_{t+1}\beta + \mu_{t+1}\}]^{-1}[Y_t - h_t \exp\{x_{t+1}\beta + \mu_{t+1}\}] \tag{5.1}$$

Let

$$Z_{t+1} = \ln\left(Y_{t+1} + \frac{1}{2}\right) - \ln h_{t+1} - x_{t+1}\beta. \tag{5.2}$$

Using the first term of the Taylor expansion (5.1)

$$Z_{t+1} \approx \mu_{t+1} = \alpha\mu_t + \omega_{t+1} \approx \alpha m_t + \omega_{t+1}. \tag{5.3}$$

Hence, given $[Y_0, Y_1, \ldots, Y_t]$, the distribution of Z_{t+1} has approximate mean αm_t and variance W. We will approximate the distribution of Z_{t+1} with a normal distribution having mean αm_t and variance W. A ln–likelihood function under this approximation is (up to addition of constants)

$$l(\alpha, W) = \sum_{t=1}^{T-1} -\frac{1}{2}\ln W - \frac{1}{2}(z_{t+1} - \alpha m_t)^2 \frac{1}{W}; \tag{5.4}$$

$$\frac{\partial l}{\partial \alpha} = \sum_{t=1}^{T-1} (z_{t+1} - \alpha m_t) m_t \frac{1}{W} = 0; \tag{5.5}$$

$$\frac{\partial l}{\partial W} = \sum_{t=1}^{T-1} -\frac{1}{2}\frac{1}{W} + \frac{1}{2}(z_{t+1} - \alpha m_t)^2 \frac{1}{W^2} = 0. \tag{5.6}$$

Thus,

$$\hat{\alpha}_L(T) = \frac{\sum_{t=1}^{T-1} (z_{t+1} m_t)}{\sum_{t=1}^{T-1} m_t^2} \tag{5.7}$$

$$\hat{W}_L(T) = \frac{1}{T-1} \sum_{t=1}^{T-1} (z_{t+1} - \hat{\alpha}_L(T) m_t)^2 \tag{5.8}$$

are the resulting estimates of α and W.

Simulation experiments suggest that $\hat{\alpha}_L$ and $\hat{W}_L$ tend to have a large positive bias.

Another possibility is to estimate α by

$$\hat{\alpha}_m(T) = \frac{1}{T} \sum_{t=1}^{T-1} m_{t+1} m_t \Big/ \sum_{t=1}^{T-1} m_t^2 \tag{5.9}$$

as in (3.7) of Section 3 and W by

$$\hat{W}_{m,L}(T) = \frac{1}{T-1}\sum_{t=1}^{T-1}(z_{t+1} - \hat{\alpha}_m(T)m_t)^2. \qquad (5.10)$$

These estimates are easy to compute. Simulation experiments suggest that this estimate of $\hat{\alpha}_m$ tends to have a negative bias and $\hat{W}_{m,L}$ has a positive bias but not as large as that of $\hat{W}_L$

6. RESULTS FROM SIMULATION EXPERIMENTS

In this section we report results of simulation experiments concerning the behavior of the approximate Kalman filter of Section 2 under various procedures for estimating α and W.

Each replication of the simulation consists of generating a Poisson time series $\{y_t; t = 0, 1, 2, \ldots, T\}$ using equations (1.1) – (1.2). The random numbers were generated using LLRANDOM II, cf. Lewis et al. (1981). If $\alpha < 1$, then μ_0 is drawn from the stationary distribution of $\{\mu_t; t \geq 0\}$. The Kalman procedure with each of the procedures for estimating α and W is then applied to $\{y_t; t = 0, 1, 2, \ldots, T\}$. For each time t, the estimate

$$\hat{\lambda}_t = \exp\left\{h_t \exp\left\{\hat{\mu}_t + x_t\hat{\beta}\right\}\right\} \qquad (6.1)$$

is computed and the mean square prediction error

$$\hat{e} = \frac{1}{T-1}\sum_{t=1}^{T-1}\left(y_{t+1} - \hat{\lambda}_t\right)^2 \qquad (6.2)$$

is calculated. In addition the average estimated values of α and W and β are computed:

$$m(\alpha) = \frac{1}{T}\sum_{t=1}^{T}\hat{\alpha}_t \qquad (6.3)$$

$$m(W) = \frac{1}{T}\sum_{t=1}^{T} \hat{W}_t \tag{6.4}$$

$$m(\beta) = \frac{1}{T}\sum_{t=1}^{T} b_t \tag{6.5}$$

where $\hat{\alpha}_t$ and $\hat{W}_t$ are the estimates of α and W obtained after the observation at time t and b_t is as in Section 1.

The simulation is replicated N times. If $\hat{e}_i, m_i(\alpha), m_i(W), m_i(\beta)$ represent the summary statistics from the i^{th} replication, then the mean of the summary statistics are computed for the N replications; i.e.

$$\bar{e} = \frac{1}{N}\sum_{i=1}^{N} \hat{e}_i; \tag{6.5}$$

$$\sqrt{\mathrm{var}[e]} = \frac{1}{N-1}\sum_{i=1}^{N} (\hat{e}_i - \bar{e})^2; \tag{6.6}$$

$$\bar{\alpha} = \frac{1}{N}\sum_{i=1}^{N} m_i(\alpha); \tag{6.7}$$

$$\overline{W} = \frac{1}{N}\sum_{i=1}^{N} m_i(W); \tag{6.8}$$

$$\bar{\beta} = \frac{1}{N}\sum_{i=1}^{N} m_i(\beta). \tag{6.8}$$

These latter statistics are then compared for the different procedures of estimating α and W.

Results for the following procedures for estimating α and W are reported.

1. DD: α and W are both known and are not estimated.

2. FT: Both α and W are estimated by a two-dimensional search of the ln-likelihood based on the Freeman-Tukey transformation (4.9). The grid for W is [0.1,3] in increments of 0.1; the grid for α is [–3,3] in increments of 0.1.

3. M/FT: α is estimated by the moment estimator using (3.7); W is estimated by searching the one-dimensional likelihood based on the Freeman-Tukey transformation with $\hat{\alpha}$ set equal to (3.7); the grid is [0.1,3] with steps of size 0.1.

4. M/LN: α is estimated using the moment estimate of (3.7). W is estimated using the moment estimator (5.10) based on the logarithmic transformation.

5. LN/LN: α and W are estimated using the moment estimators (5.9) and (5.10) based on the logarithmic transformation.

All the procedures have a lower bound of –2 on $\hat{\mu}_t$, a lower bound of 0.1 on $\hat{W}_t$, and an upper bound of 1 on the absolute value of $\hat{\alpha}_t$.

For the results of the simulation experiments reported in Table 1, $\alpha = 0.5$, $W = 0.25$, $x_t \equiv 1$, $h_t \equiv 1$, $\beta = 0.5$. The initial values of the estimates $\hat{W}_0 = 0.25$, $\hat{\alpha}_0 = 0.5$, $\hat{\mu}_0 = 0$; for the Kalman $\hat{C}_0 = 1$, $\hat{\tau}_0 = 1$, $\hat{\rho}_0 = 0$, $\hat{\beta} = 0$. The two-dimensional search for the estimates of α and W in FT takes a large amount of computer time. As a result, the number of replications for the experiments reported in Table 1 is small. However, the results of Table 1 suggest that the two most promising procedures are M/LN and M/FT.

In Table 2, the number of replications is 100. The procedures M/LN and M/FT only are compared. In Table 2, the initial values of $\hat{W}_0 = 0.25$, $\hat{\alpha}_0 = 0.5$, $\hat{\mu}_0 = 0$; for the Kalman $\hat{C}_0 = 1$, $\hat{\tau}_0 = 1$, $\hat{\rho}_0 = 0$, $\hat{\beta} = 0$. .

In Table 3, $\{x_t\} = \{0.25, 0.5, 1, 0.25, 0.5, 1, \ldots\}$, $\beta = 0.5$, $W = 0.25$, $\alpha = 0.5$. The initial values of $\hat{W}_0 = 0.25$, $\hat{\alpha}_0 = 0.5$, $\hat{\mu}_0 = 0$; for the Kalman $\hat{C}_0 = 1$, $\hat{\tau}_0 = 1$, $\hat{\rho}_0 = 0$, $\hat{\beta} = 0$.

In Table 4, the parameters are the same as in Table 3 except $W = 1$. The initial values of $\hat{W}_0 = 0.25$, $\hat{\alpha}_0 = 0.5$, $\hat{\mu}_0 = 0$; for the Kalman $\hat{C}_0 = 1$, $\hat{\tau}_0 = 1$, $\hat{\rho}_0 = 0$, $\hat{\beta} = 0$.

Comparison of Tables 2 and 3 suggests, not surprisingly, that the mean square prediction error is smaller when there is a variable covariate $\{x_t\}$.

Comparison of Tables 3 and 4 suggests, not surprisingly, that a larger value of W results in a larger mean square prediction error.

TABLE 1.

SERIES LENGTH = 5; 20 REPLICATIONS

METHOD:	DD	FT	M/FT	M/LN	LN/LN
Mean MSE	4.34	4.91	4.77	4.19	4.96
St. Dev. MSE	3.21	3.82	3.64	3.17	3.76
Max MSE	11.0	13.2	11.7	12.5	13.5
Mean $\hat{\alpha}$	–	0.10	0.11	0.15	0.26
Mean $\hat{W}$	–	0.48	0.67	0.98	0.77

SERIES LENGTH = 10; 20 REPLICATIONS

METHOD:	DD	FT	M/FT	M/LN	LN/LN
Mean MSE	2.94	3.15	3.08	2.77	3.36
St. Dev. MSE	2.27	2.42	2.28	1.93	2.54
Max MSE	10.4	11.1	10.1	8.27	11.2
Mean $\hat{\alpha}$	–	0.12	0.08	0.09	0.19
Mean $\hat{W}$	–	0.23	0.32	0.78	0.69

SERIES LENGTH = 20; 40 REPLICATIONS

METHOD:	DD	FT	M/FT	M/LN	LN/LN
Mean MSE	3.38	3.90	3.72	3.29	3.97
St. Dev. MSE	2.00	2.81	2.43	2.16	2.62
Max MSE	10.7	13.5	12.1	10.3	11.2
Mean $\hat{\alpha}$	–	0.33	0.15	0.17	0.31
Mean $\hat{W}$	–	0.22	0.31	0.79	0.73

TABLE 2. $x_t \equiv 1$, $\alpha = 0.5$, $W = 0.25$

SERIES LENGTH = 5; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	4.21	4.75	4.32
St. Dev. MSE	4.68	5.81	5.40
Max MSE	28.7	38.0	34.1
Mean $\hat{\alpha}$	–	0.08	0.09
Mean $\hat{W}$	–	0.55	0.93
Mean $\hat{\beta}$	0.34	0.25	0.23

SERIES LENGTH = 10; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	3.98	4.41	4.14
St. Dev. MSE	3.27	3.87	3.58
Max MSE	16.4	19.0	17.3
Mean $\hat{\alpha}$	–	0.07	0.10
Mean $\hat{W}$	–	0.45	0.88
Mean $\hat{\beta}$	0.42	0.36	0.31

SERIES LENGTH = 20; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	3.96	4.35	4.18
St. Dev. MSE	2.59	3.30	3.06
Max MSE	20.3	26.5	23.6
Mean $\hat{\alpha}$	–	0.10	0.13
Mean $\hat{W}$	–	0.41	0.82
Mean $\hat{\beta}$	0.48	0.43	0.37

TABLE 3. VARIABLE $\{x_t\}$, $\alpha = 0.5$, $W = 0.25$

SERIES LENGTH = 5; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	3.69	3.89	3.70
St. Dev. MSE	3.83	4.09	4.00
Max MSE	15.6	18.4	19.9
Mean $\hat{\alpha}$	–	0.13	0.12
Mean $\hat{W}$	–	0.52	0.80
Mean $\hat{\beta}$	0.32	0.21	0.18

SERIES LENGTH = 20; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	3.11	3.32	3.21
St. Dev. MSE	2.48	2.80	2.68
Max MSE	20.4	22.7	20.2
Mean $\hat{\alpha}$	–	0.12	0.10
Mean $\hat{W}$	–	0.35	0.78
Mean $\hat{\beta}$	0.41	0.36	0.28

TABLE 4. VARIABLE $\{x_t\}$, $\alpha = 0.5$, $W = 1$

SERIES LENGTH = 5; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	19.21	20.70	19.88
St. Dev. MSE	52.2	53.0	52.1
Max MSE	415.1	390.1	393.5
Mean $\hat{\alpha}$	–	0.19	0.18
Mean $\hat{W}$	–	0.93	1.05
Mean $\hat{\beta}$	0.31	0.14	0.12

SERIES LENGTH = 20; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	18.93	21.6	20.4
St. Dev. MSE	28.6	31.5	30.9
Max MSE	181.7	192.1	188.5
Mean $\hat{\alpha}$	–	0.20	0.19
Mean $\hat{W}$	–	1.2	1.1
Mean $\hat{\beta}$	0.55	0.32	0.31

Of the two procedures, M/LN tends to have the smaller mean MSE. M/LN tends to have a positive bias estimating W and negative biases estimating $\hat{\alpha}$ and $\hat{\beta}$. The procedure M/FT takes more computational effort, tends to underestimate W, α and β and produces slightly higher mean MSE.

Tables 5-7 report results comparing procedures estimating α and W with the procedure NFT in which both α and W are estimated using the Newton procedure of Section 4. For all the procedures reported in these tables, there is no bounding on $\hat{\alpha}$. In Table 5, $x_t \equiv 1$, $h_t \equiv 1$, $\beta = 0.5$, $\alpha = 0.5$, $W = 0.25$. In Table 6, $x_t = 0.25, 0.5, 1, 0.25, 0.5, 1, \ldots$, and the other parameters are as before. For Tables 5 and 6 the initial values of $\hat{W}_0 = 0.25$, $\hat{\alpha}_0 = 0.5$, $\hat{\mu}_0 = 0$, $\hat{C}_0 = 1$, $\hat{\tau}_0 = 1$, $\hat{\rho}_0 = 0$. The values in parentheses are estimates of standard deviations; e.g.

$$\left[\frac{1}{N-1}\sum_{i=1}^{N}(m_i(\alpha) - \bar{\alpha})^2\right]^{\frac{1}{2}}.$$

For Table 7, $\{x_t\}$ is as in Table 6, $\alpha = 0.5$, $W = 0.25$, $\beta_0 = 0.5$, and the initial estimates are $\hat{\alpha}_0 = 0$, $\hat{W}_0 = 1$, $\hat{C}_0 = 1$, $\hat{\tau}_0 = 1$, $\hat{\rho}_0 = 0$, $\hat{\mu}_0 = 0$.

The Newton procedure of Section 4 does not take as much time as the two-dimensional search used in Tables 1-2; however, it is still a much larger computational effort than the other two procedures. Once again, based on the mean MSE and computational effort the procedure M/LN appears to be the most attractive. The procedure M/LN once again appears to overestimate W and underestimate α and β.

TABLE 5. $x_t \equiv 1$; $\alpha = 0.5$, $W = 0.25$, $\beta = 0.5$

SERIES LENGTH = 5, 25 REPLICATIONS

METHOD:	DD	NFT	M/FT	M/LN
Mean MSE	3.94	4.44	4.32	3.89
Std Dev. MSE	3.08	3.66	3.49	3.10
Max MSE	11.0	13.2	11.7	12.5
Mean $\hat{\alpha}$	-	0.06 (0.37)	0.08 (0.22)	0.10 (0.23)
Mean $\hat{W}$	-	0.46 (0.47)	0.60 (0.60)	0.93 (0.45)
Mean $\hat{\beta}$	0.28	0.18	0.21	0.18

SERIES LENGTH = 20, 25 REPLICATIONS

METHOD:	DD	NFT	M/FT	M/LN
Mean MSE	3.11	3.40	3.32	3.14
Std Dev. MSE	1.83	2.12	2.11	2.03
Max MSE	8.50	9.29	9.60	8.78
Mean $\hat{\alpha}$	-	0.22 (0.39)	0.10 (0.30)	0.12 (0.29)
Mean $\hat{W}$	-	0.29 (0.24)	0.32 (0.26)	0.80 (0.23)
Mean $\hat{\beta}$	0.38	0.23	0.31	0.23

TABLE 6. VARIABLE $\{x_t\}$; $\alpha = 0.5$, $W = 0.25$, $\beta = 0.5$

SERIES LENGTH = 20, 25 REPLICATIONS

METHOD:	DD	NFT	M/FT	M/LN
Mean MSE	2.63	2.77	2.81	2.63
Std Dev. MSE	1.78	1.95	2.06	1.79
Max MSE	9.24	10.10	10.86	9.12
Mean $\hat{\alpha}$	-	0.21 (0.38)	0.06 (0.32)	0.06 (0.34)
Mean $\hat{W}$	-	0.20 (0.14)	0.25 (0.20)	0.68 (0.18)
Mean $\hat{\beta}$	0.33	0.21	0.27	0.18

TABLE 7. VARIABLE $\{x_t\}$; $\alpha = 0.5$, $W = 0.25$, $\beta = 0.5$

$\hat{\alpha}_0 = 0$, $\hat{W}_0 = 1$, $\hat{C}_0 = 1$, $\hat{\tau}_0 = 1$, $\hat{\rho}_0 = 0$, $\hat{\mu}_0 = 0$

SERIES LENGTH = 20, 100 REPLICATIONS

METHOD:	DD	NFT	M/FT	M/LN
Mean MSE	2.95	3.25	3.21	3.04
Std Dev. MSE	2.21	2.70	2.67	2.28
Max MSE	13.2	16.5	16.5	13.8
Mean $\hat{\alpha}$	-	0.12 (0.34)	0.11 (0.28)	0.09 (0.29)
Mean $\hat{W}$	-	0.34 (0.44)	0.36 (0.32)	0.73 (0.19)
Mean $\hat{\beta}$	0.36	0.25	0.28	0.21

7. SUMMARY AND CONCLUSIONS

In this paper we have investigated approximate methods for estimation and prediction for the hierarchical Poisson/normal time series model given in (1.1)–(1.2). For given values of the random walk parameters, α and W, the joint distribution of $(\boldsymbol{\beta}, \mu_t)$ is approximated by a bivariate normal distribution using the Laplace method. Various heuristic methods for estimating α and W are presented. Based on simulation results and ease of computation, it is recommended that α be estimated by (3.7)

$$\hat{\alpha}(T) = \sum_{t=1}^{T} m_t m_{t-1} \Big/ \sum_{t=1}^{T} m_{t-1}^2$$

and W be estimated by (5.10)

$$\hat{W}(T) = \frac{1}{T-1} \sum_{t=1}^{T-1} (z_{t+1} - \hat{\alpha}(T) m_t)^2$$

where z_{t+1} is given by (5.2).

These estimates of α and W along with the approximate Kalman procedure using the Laplace method provide a computationally easy procedure for prediction of the time series.

Figures 4-7 show results of a simulation of model (1.1)–(1.2) for $t = 1, \ldots, 100$. In the example $\beta = 0.5$, $\{x_t\}$ takes the values $\{0.25, 0.5, 1\}$ over and over, and $h_t \equiv 1$. $\{\omega_t\}$ are iid normal with mean 0, variance 0.25, and $\alpha = 0.5$. The simulation starts with μ_0 drawn from the stationary distribution of $\{\mu_t\}$, a normal distribution with mean 0 and variance $W/(1-\alpha^2) = \frac{1}{3}$. The values of α and W are estimated using (3.7) and (5.10). Initial values of α and W are

$\alpha_0 = 0.5$ and $W = 1$, $C_0 = 1$, $\tau_0 = 1$, $\rho_0 = 0$, $\beta_0 = 0$, $\mu_0 = 0$. There are no bounds on $\hat{\mu}_t$, $\hat{\alpha}$ and $\hat{W}$.

Figure 4 displays the count data. Also displayed are the true λ_t (circles), the estimated λ_t when both α and W are known (dashed line) and the estimated λ_t when both α and W are estimated (solid line). The $\hat{\lambda}_t$ when both α and W are estimated is perhaps a little more responsive to changes in the data than when α and W are known. The difference between the two estimates of λ_t is greatest for small times t.

Figure 5 presents plots of the estimated $\sqrt{W}$ and α. Note that W has a positive bias which will make the Kalman procedure more responsive to the count data. The estimated values of α appear to be reasonable.

Figure 6 presents estimates of β and its estimated standard deviation $\sqrt{\tau_t}$ both for the Kalman with α and W known (dotted line) and with α and W unknown (solid line). The estimates of τ_t generally decrease as t increases reflecting the model assumption that β is constant. The estimates of β appear to be reasonable. The estimates of β with α and W also estimated tend to be smaller than those for which α and W are known; this behavior may be due to the fact that estimation of α and W is accounting for some of the variability of the data that would otherwise be accounted for by β.

Figure 7 displays the true μ_t (circles), the estimated $\hat{\mu}_t = m_t$ with parameters α and W known (dotted line) and the estimated $\hat{\mu}_t$ with parameters α and W estimated (solid line). The count data are also displayed. The estimates of $\hat{\mu}_t$ with α and W estimated are more variable than those when α and W are known reflecting the greater responsiveness of the Kalman to the

data when the estimated W has a position bias. Once again the estimates of μ_t appear to be reasonable.

REFERENCES

Bishop,Y.M.M., Feinberg, S. E. and Holland, P. W., *Discrete Multivariate Analysis: Theory and Practice*, MIT Press, Cambridge, MA, 1975.

Carlin, B. P., Polson, N. G. and Stoffer, D. S., "A Monte Carlo approach to nonnormal and nonlinear state space modeling," To appear *J. Amer. Statist. Assoc.*

Cox, D. R. and Hinkley, D. W., *Theoretical Statistics*, Chapman and Hall, 1974.

Cox, D. R. and Lewis, P. A. W., *The Statistical Analysis of Series of Events*, Methuen, London, 1968.

de Bruijn, N. G., *Asymptotic Methods in Analysis*, Interscience, New York, 1958.

Delampady, M., Yee, I. and Zidek, J. V., "Bayesian approach to a problem in discrete time series" (abstract) *The Institute of Mathematical Statistics Bulletin* **20**, (1991), p. 308.

Easton, G. S., "Location compromise maximum likelihood estimators" in *Configural Polysampling*, ed. S. Morgenthaler and J. W. Tukey, Wiley, New York, 1991, pp. 157-192.

Feller, W., *An Introduction to Probability Theory and its Applications*, Vol. II, Wiley, New York, 1966.

Freeman, M.F. and Tukey, J. W., "Transformations related to the angular and the square root," *Ann. Math. Stat.*, **21** (1950), pp. 607-611.

Freeman, M.F. and Tukey, J. W., "Transformations related to the angular and the square root." *Memorandum Report 24*, Statistical Research Group, Princeton, 1949.

Harvey, A. C. and Fernandes, C., "Time series models for count or qualitative observations, (with discussion), *J. Business and Econ. Statist.*, **7**, (1989), pp. 407-422.

Lambert, D. "Taking roots of count data." *AT&T Statistical Research Report*, No. 88, November 1989.

Lewis, P. A. W., and Uribe, L., *The new Naval Postgraduate School random number generator—LLRANDOM II*, Naval Postgraduate School Technical Report NPS 55-81-005, Monterey, CA, 1981.

West, M. and Harrison, J., *Bayesian Forecasting and Dynamic Models*, Springer-Verlag, New York, 1989.

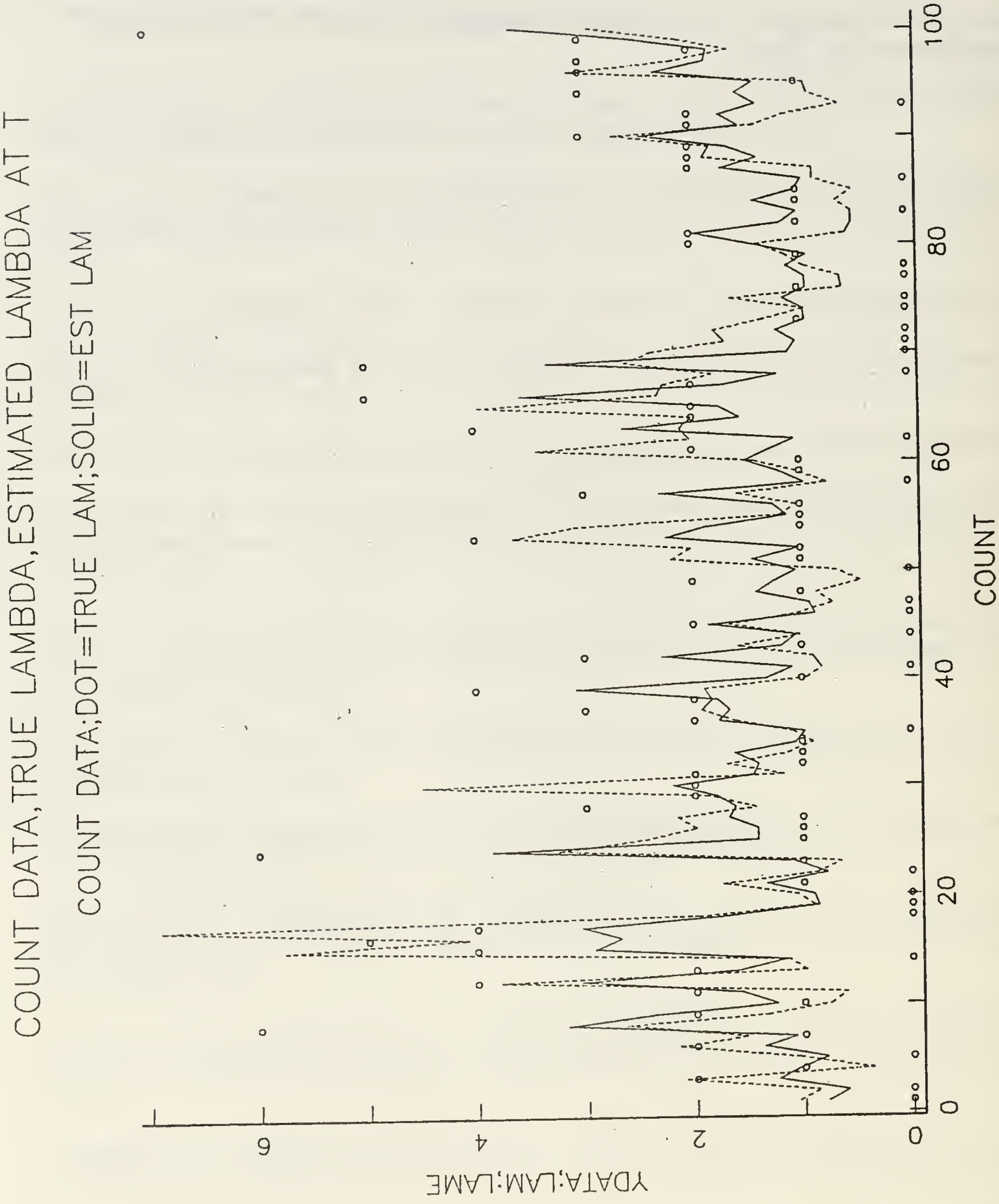

COUNT DATA,TRUE LAMBDA,ESTIMATED LAMBDA AT T
COUNT DATA;DOT=TRUE LAM;SOLID=EST LAM
YDATA;LAM;LAME
0
2
4
6
0
20
40
60
80
100
COUNT

Figure 1

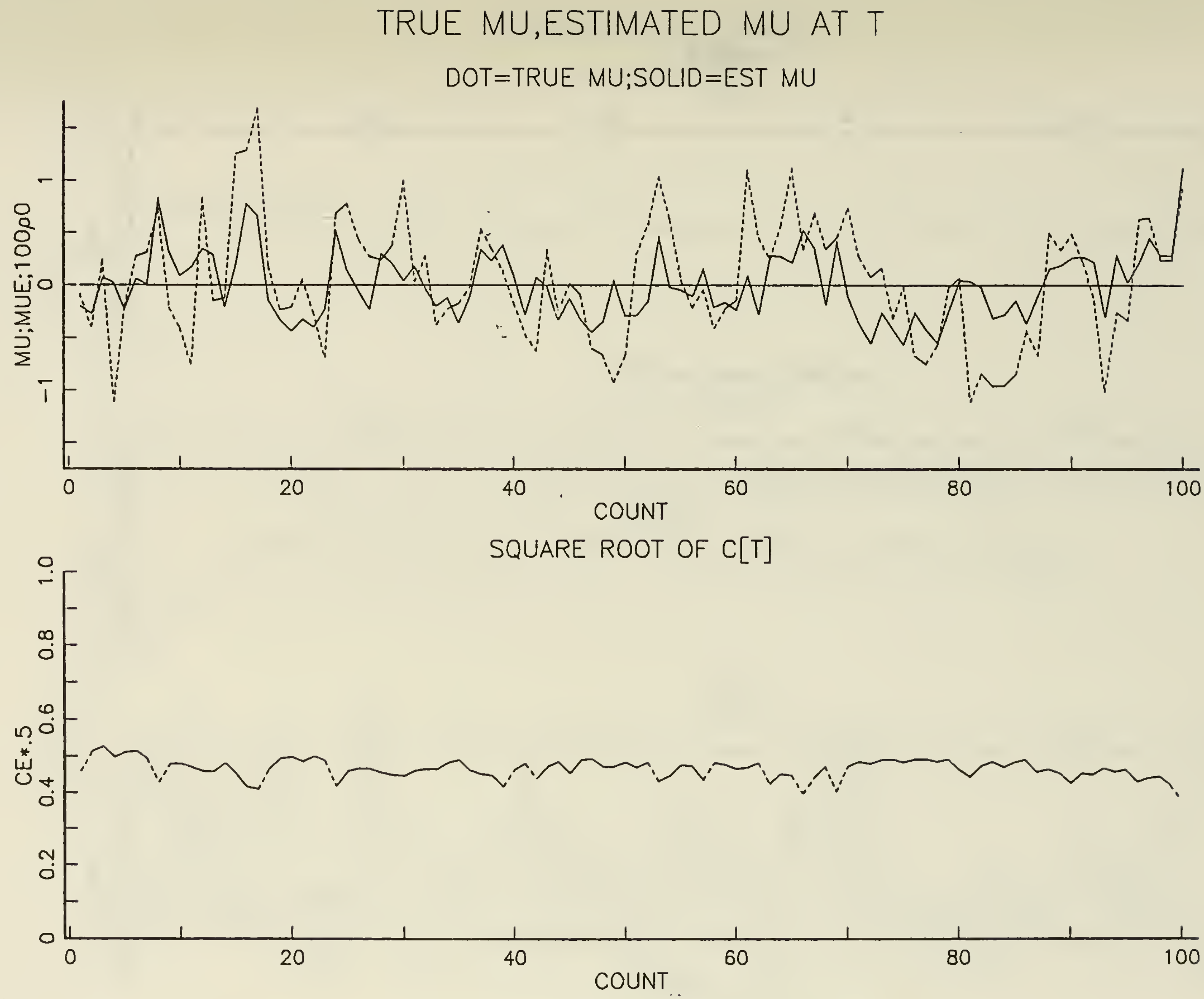

TRUE MU,ESTIMATED MU AT T
DOT=TRUE MU;SOLID=EST MU
MU;MUE;100ρ0
1
0
−1
0
20
40
60
80
100
COUNT
SQUARE ROOT OF C[T]
CE*.5
1.0
0.8
0.6
0.4
0.2
0
0
20
40
60
80
100
COUNT

Figure 2

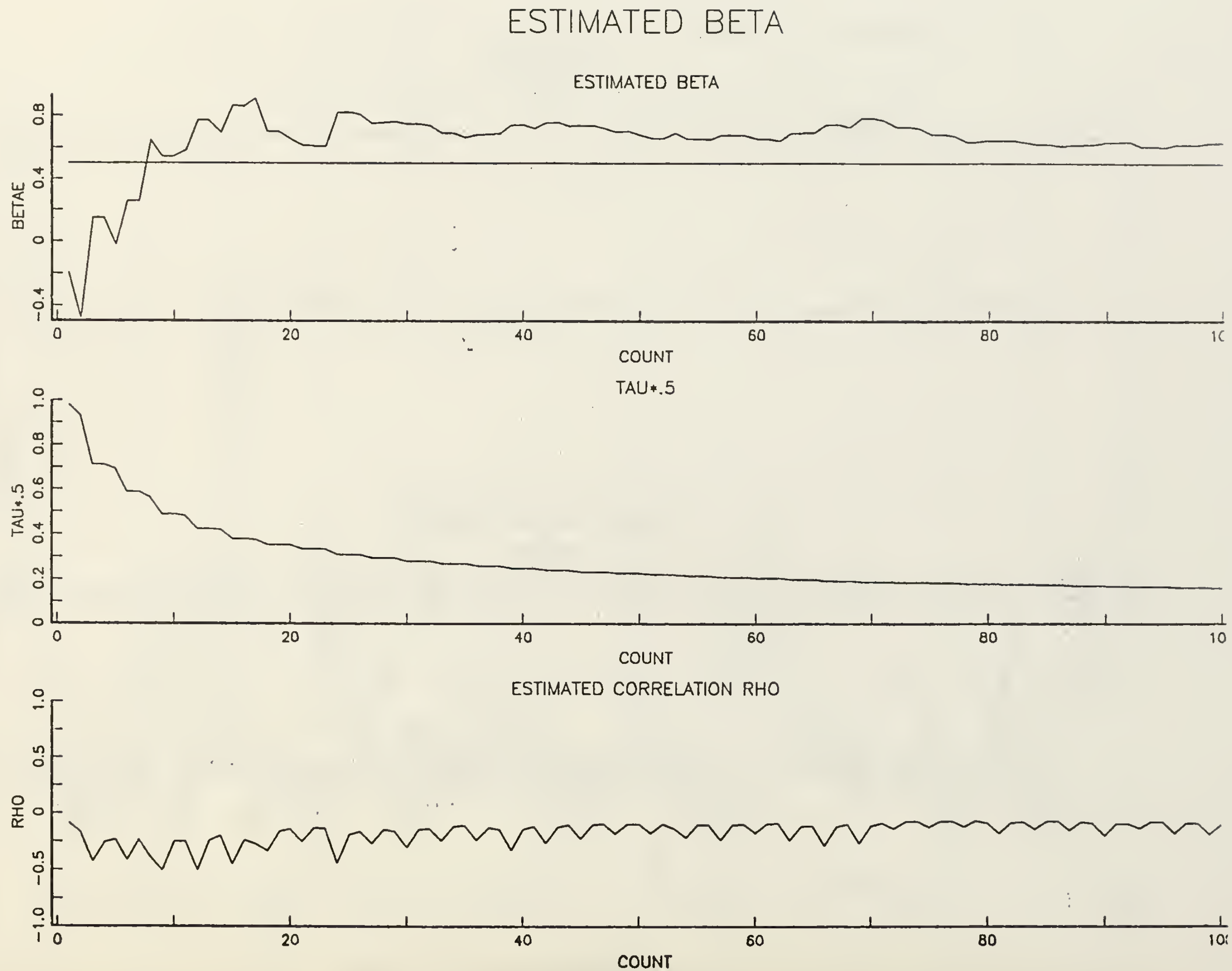
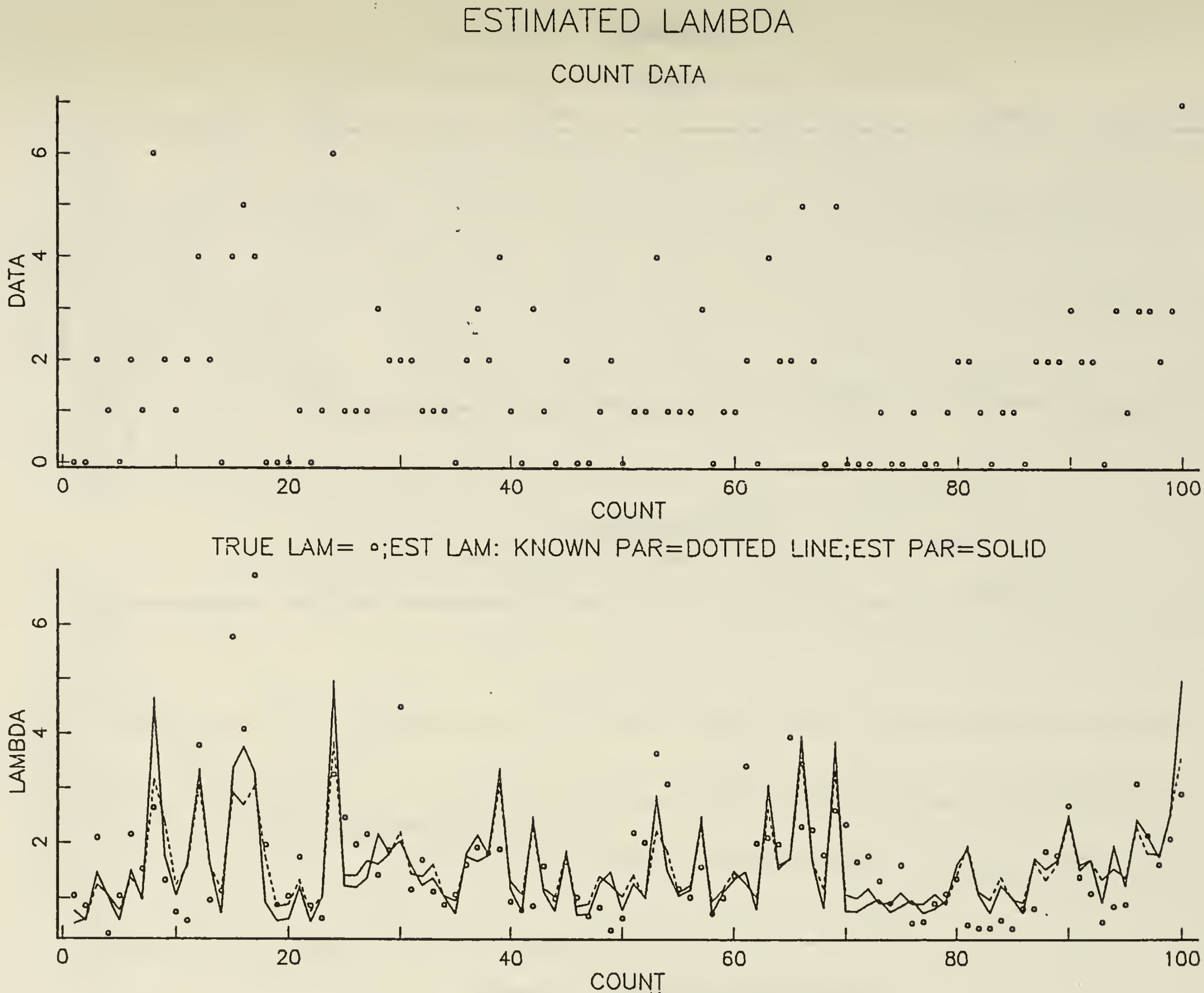

ESTIMATED BETA
ESTIMATED BETA
BETAE
0.8
0.4
0
−0.4
0
20
40
60
80
1(
COUNT
TAU*.5
TAU*.5
1.0
0.8
0.6
0.4
0.2
0
0
20
40
60
80
10
COUNT
ESTIMATED CORRELATION RHO
RHO
1.0
0.5
0
−0.5
−1.0
0
20
40
60
80
10(
COUNT

Figure 3 .



ESTIMATED LAMBDA
COUNT DATA
DATA
COUNT
TRUE LAM= o;EST LAM: KNOWN PAR=DOTTED LINE;EST PAR=SOLID
LAMBDA
COUNT

Figure 4

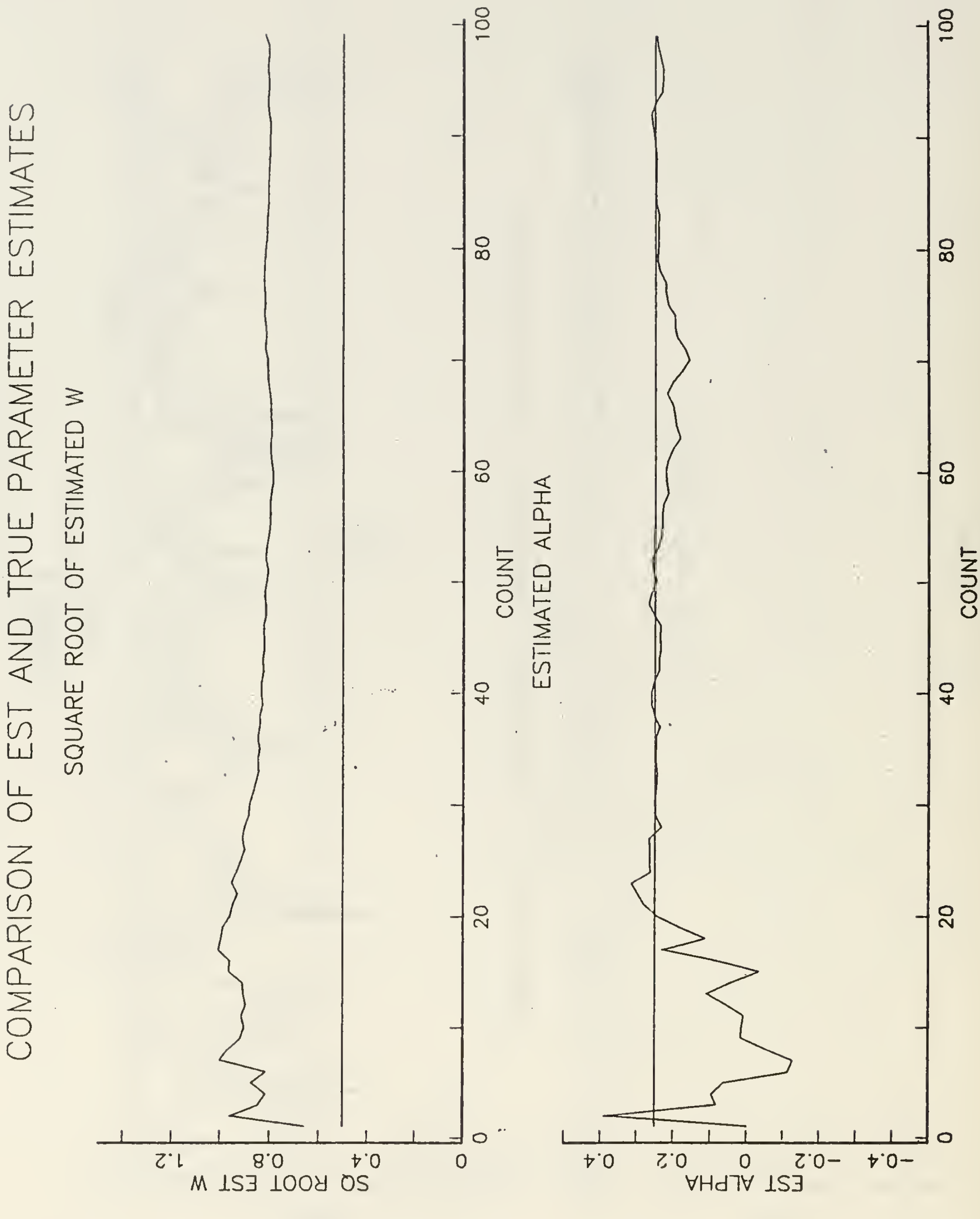
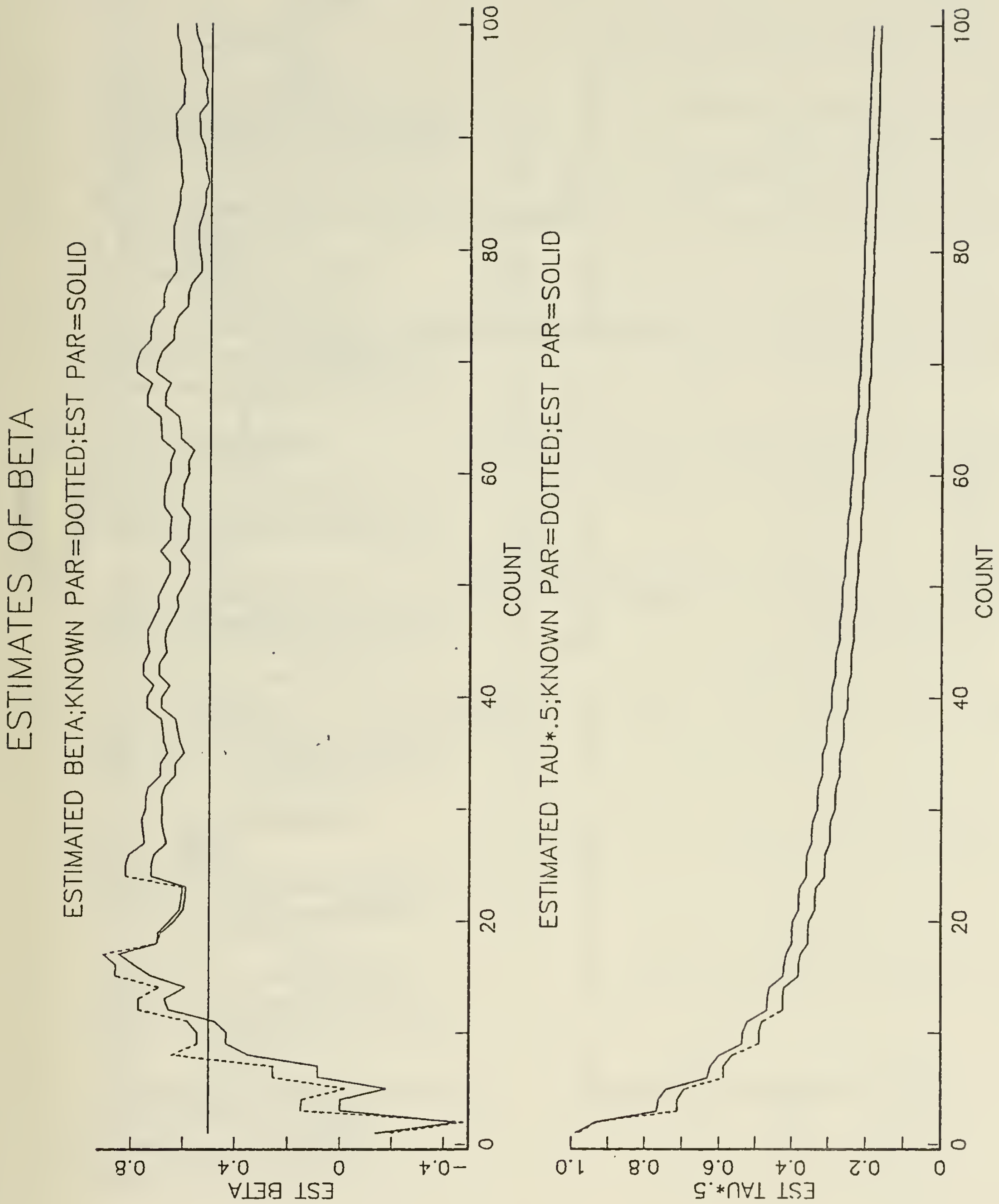

COMPARISON OF EST AND TRUE PARAMETER ESTIMATES
SQUARE ROOT OF ESTIMATED W
SQ ROOT EST W
0
0.4
0.8
1.2
0
20
40
60
80
100
COUNT
ESTIMATED ALPHA
EST ALPHA
−0.4
−0.2
0
0.2
0.4
0
20
40
60
80
100
COUNT

Figure 5



ESTIMATES OF BETA
ESTIMATED BETA;KNOWN PAR=DOTTED;EST PAR=SOLID
EST BETA
0.8
0.4
0
-0.4
0
20
40
60
80
100
COUNT
ESTIMATED TAU*.5;KNOWN PAR=DOTTED;EST PAR=SOLID
EST TAU*.5
1.0
0.8
0.6
0.4
0.2
0
0
20
40
60
80
100
COUNT

Figure 6

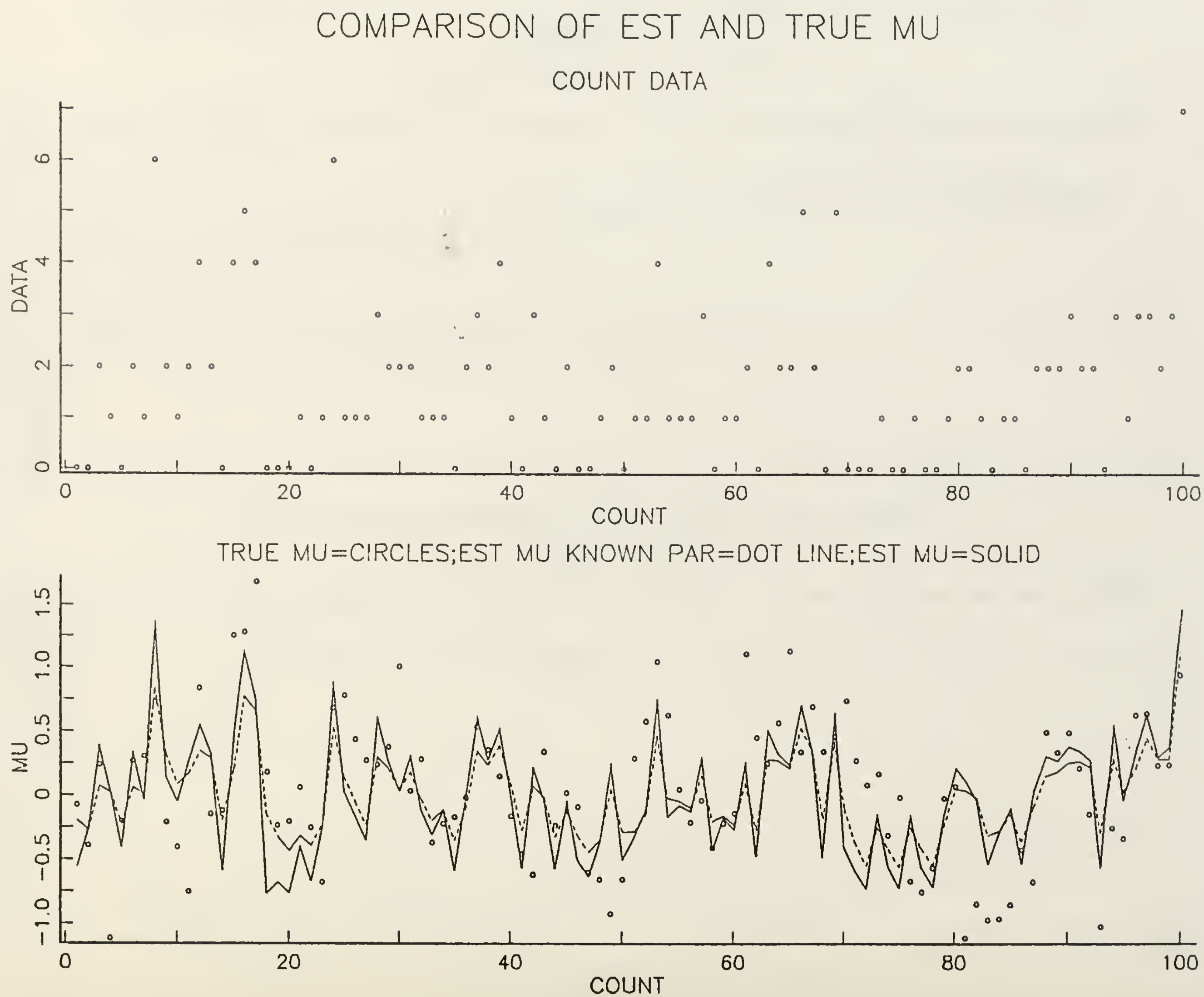

COMPARISON OF EST AND TRUE MU
COUNT DATA
DATA
COUNT
TRUE MU=CIRCLES;EST MU KNOWN PAR=DOT LINE;EST MU=SOLID
MU
COUNT

Figure 7

INITIAL DISTRIBUTION LIST

1. Library (Code 0142) 2
   Naval Postgraduate School
   Monterey, CA 93943-5000

2. Defense Technical Information Center 2
   Cameron Station
   Alexandria, VA 22314

3. Office of Research Administration (Code 012) 1
   Naval Postgraduate School
   Monterey, CA 93943-5000

4. Prof. Peter Purdue 1
   Code OR-Pd
   Naval Postgraduate School
   Monterey, CA 93943-5000

5. Department of Operations Research (Code 55) 1
   Naval Postgraduate School
   Monterey, CA 93943-5000

6. Prof. Donald Gaver, Code OR-Gv 15
   Naval Postgraduate School
   Monterey, CA 93943-5000

7. Prof. Patricia Jacobs 15
   Code OR/Jc
   Naval Postgraduate School
   Monterey, CA 93943-5000

8. Center for Naval Analyses 1
   4401 Ford Avenue
   Alexandria, VA 22302-0268

9. Dr. David Brillinger 1
   Statistics Department
   University of California
   Berkeley, CA 94720

10. Dr. R. Gnanadesikan...1
Bellcore
445 South Street
Morris Township NJ 07962-1910

11. Prof. Bernard Harris...1
Dept. of Statistics
University of Wisconsin
610 Walnut Street
Madison, WI 53706

12. Prof. I. R. Savage...1
Dept. of Statistics
Yale University
New Haven, CT 06520

13. Prof. W. R. Schucany...1
Dept. of Statistics
Southern Methodist University
Dallas, TX 75222

14. Prof. D. C. Siegmund...1
Dept. of Statistics
Sequoia Hall
Stanford University
Stanford, CA 94305

15. Prof. H. Solomon...1
Department of Statistics
Sequoia Hall
Stanford University
Stanford, CA 94305

16. Dr. Ed Wegman...1
George Mason University
Fairfax, VA 22030

17. Dr. P. Welch...1
IBM Research Laboratory
Yorktown Heights, NY 10598

18. Dr. Neil Gerr...1
Office of Naval Research
Arlington, VA 22217

19. Prof. Roy Welsch 1
Sloan School
M.I.T.
Cambridge, MA 02139

20. Dr. J. Abrahams, Code 1111, Room 607 1
Mathematical Sciences Division, Office of Naval Research
800 North Quincy Street
Arlington, VA 22217-5000

21. Prof. J. R. Thompson..1
Dept. of Mathematical Science
Rice University
Houston, TX 77001

22. Dr. P. Heidelberger 1
IBM Research Laboratory
Yorktown Heights
New York, NY 10598

23. Prof. M. Leadbetter..1
Department of Statistics
University of North Carolina
Chapel Hill, NC 27514

24. Prof. D. L. Iglehart..1
Dept. of Operations Research
Stanford University
Stanford, CA 94350

25. Prof. J. B. Kadane..1
Dept. of Statistics
Carnegie-Mellon University
Pittsburgh, PA 15213

26. Prof. J. Lehoczky 1
Department of Statistics
Carnegie-Mellon University
Pittsburgh, PA 15213

28. Prof. M. Mazumdar..1
Dept. of Industrial Engineering
University of Pittsburgh
Pittsburgh, PA 15235

29. Prof. M. Rosenblatt 1
Department of Mathematics
University of California, San Diego
La Jolla, CA 92093

30. Prof. H. Chernoff 1
Department of Statistics
Harvard University
1 Oxford Street
Cambridge, MA 02138

31. Dr. T. J. Ott 1
Bellcore,
445 South Street
Morristown, NJ 07962-1910
(MRE 2P388)

32. Dr. Alan Weiss 1
AT&T Bell Laboratories
Mountain Avenue
Murray Hill, NJ 07974

33. Prof. Joseph R. Gani 1
Mathematics Department
University of California
Santa Barbara, CA 93106

34. Prof. Frank Samaniego 1
Statistics Department
University of California
Davis, CA 95616

35. Dr. James McKenna 1
Bell Communications Research
445 South Street
Morristown, NJ 07960-1910

35. Dr. Shlomo Halfin 1
Bellcore,
445 South Street
Morristown, NJ 07962-1910
(MRE 2L309)

36. Commander 1
Attn: G. F. Kramer
Naval Ocean Systems Center
Code 421
San Diego, CA 92152-5000

37. Prof. Tom A. Louis 1
School of Public Health
University of Minnesota
Mayo Bldg. A460
Minneapolis, MN 55455

38. Dr. Nan Laird..1
Biostatistics Dept.
Harvard School of Public Health
677 Huntington Ave.
Boston, MA 02115

39. Dr. Marvin Zelen..1
Biostatistics Department
Harvard School of Public Health
677 Huntington Ave.
Boston, MA 02115

40. Dr. John Orav 1
Biostatistics Department
Harvard School of Public Health
677 Huntington Ave.
Boston, MA 02115

41. Prof. R. Douglas Martin..1
Department of Statistics, GN-22
University of Washington
Seattle, WA 98195

42. Prof. W. Stuetzle..1
Department of Statistics
University of Washington
Seattle, WA 98195

43. Prof. F. W. Mosteller..1
Department of Statistics
Harvard University
1 Oxford St.
Cambridge, MA 02138

44. Dr. D. C. Hoaglin 1
Department of Statistics
Harvard University
1 Oxford Street
Cambridge, MA 02138

45. Prof. N. D. Singpurwalla..1
George Washington University
Washington, DC 20052

46. Prof. George S. Fishman..1
Curr. in OR & Systems Analysis
University of North Carolina
Chapel Hill, NC 20742

47. Dr. Alan F. Petty..1
Code 7930
Naval Research Laboratory
Washington, DC 20375

48. Prof. Bradley Efron..1
Statistics Dept.
Sequoia Hall
Stanford University
Stanford, CA 94305

49. Prof. Carl N. Morris..1
Statistics Department
Harvard University
1 Oxford St.
Cambridge, MA 02138

50. Dr. John E. Rolph 1
RAND Corporation
1700 Main St.
Santa Monica, CA 90406

51. Prof. Linda V. Green 1
Graduate School of Business
Columbia University
New York, NY 10027

52. Dr. David Burman..1
AT&T Bell Telephone Laboratories
600 Mountain Avenue
Murray Hill, NJ 07974

53. Dr. Edward G. Coffman, Jr. 1
AT&T Bell Telephone Laboratories
600 Mountain Avenue
Murray Hill, NJ 07974

54. Prof. William Jewell 1
Operations Research Department
University of California, Berkeley
Berkeley, CA 94720

55. Prof. D. C. Siegmund 1
Dept. of Statistics, Sequoia Hall
Stanford University
Stanford, CA 94305

56. Operations Research Center, Rm E40-164 1
Massachusetts Institute of Technology
Attn: R. C. Larson and J. F. Shapiro
Cambridge, MA 02139

57. Arthur P. Hurter, Jr. 1
Professor and Chairman
Dept. of Industrial Engineering and Management Sciences
Northwestern University
Evanston, IL 60201-9990

58. Institute for Defense Analysis 1
1800 North Beauregard
Alexandria, VA 22311

59. Prof. J. W. Tukey 1
Statistics Dept., Fine Hall
Princeton University
Princeton, NJ 08540

60. Dr. Daniel H. Wagner..1
Station Square One
Paoli, PA 19301

61. Dr. Colin Mallows 1
AT&T Bell Telephone Laboratories
600 Mountain Avenue
Murray Hill, NJ 07974

62. Dr. D. Pregibon 1
AT&T Bell Telephone Laboratories
Mountain Avenue
Murray Hill, NJ 07974

63. Dr. Jon Kettenring 1
Bellcore
445 South Street
Morris Township, NJ 07962-1910

64. Prof. David L. Wallace 1
Statistics Dept., University of Chicago
5734 S. University Ave.
Chicago, IL 60637

65. Dr. S. R. Dalal 1
Bellcore
445 South Street
Morristown, NJ 07962-1910

66. Dr. M. J. Fischer 1
Defense Communications Agency
1860 Wiehle Avenue
Reston, VA 22070

67. Dr. Prabha Kumar
Defense Communications Agency 1
1860 Wiehle Avenue
Reston, VA 22070

68. Dr. B. Doshi 1
AT&T Bell Laboratories
HO 3M-335
Holmdel, NJ 07733

69. Dr. D. M. Lucantoni 1
AT&T Bell Laboratories
Holdmel, NJ 07733

70. Dr. V. Ramaswami 1
MRE 2Q-358
Bell Communications Research, Inc.
445 South Street
Morristown, NJ 07960

71. Prof. G. Shantikumar 1
The Management Science Group
School of Business Administration
University of California
Berkeley, CA 94720

72. Dr. D. F. Daley 1
Statistic Dept. (I.A.S.)
Australian National University
Canberra, A.C.T. 2606
AUSTRALIA

73. Dr. Guy Fayolle 1
I.N.R.I.A.
Dom de Voluceau-Rocquencourt
78150 Le Chesnay Cedex
FRANCE

74. Professor H. G. Daellenbach 1
Department of Operations Research
University of Canterbury
Christchurch, NEW ZEALAND

75. Koh Peng Kong 1
OA Branch, DSO
Ministry of Defense
Blk 29 Middlesex Road
SINGAPORE 1024

76. Professor Sir David Cox 1
Nuffield College
Oxford, OXI INF
ENGLAND

77. Dr. A. J. Lawrence 1
Dept. of Mathematics,
University of Birmingham
P. O. Box 363
Birmingham B15 2TT
ENGLAND

78. Dr. F. P. Kelly 1
Statistics Laboratory
16 Mill Lane
Cambridge
ENGLAND

79. Dr. R. J. Gibbens 1
Statistics Laboratory
16 Mill Lane
Cambridge
ENGLAND

80. Dr. John Copas..1
Dept. of Mathematics,
University of Birmingham
P. O. Box 363
Birmingham B15 2TT
ENGLAND

81. Dr. D. Vere-Jones..1
Dept. of Math, Victoria Univ. of Wellington
P. O. Box 196
Wellington
NEW ZEALAND

82. Prof. Guy Latouche..1
University Libre Bruxelles
C. P. 212, Blvd. De Triomphe
B-1050 Bruxelles
BELGIUM